A Fast Rotating Bose-Einstein Condensate on a Disc

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1 Zusammenfassung/Abstract

In der vorliegenden Arbeit studieren wir ein schnell rotierendes Bose-Einstein Kondensat auf einer Scheibe im Grenzwert starker Wechselwirkung (Thomas-Fermi Limes) mithilfe der 2D Gross-Pitaevskii (GP) Theorie. Wir bestimmen sowohl eine untere als auch eine obere Schranke zur Energie eines solchen Bose Gases mit Dirichlet Randbedingung. Ein Beweis dieser Schranken unter Neumann Randbedingungen wurde zuerst in den Arbeiten [CY], [CDY] geführt. Unser Beitrag besteht unter anderem darin, diese Betrachtungen durch einen zusätzlichen Faktor in der Versuchswellenfunktion auf den Fall von Dirichlet Randbedingungen zu erweitern. Dabei zeigen wir methodisch analog zu [CDY], dass in einem bestimmten Bereich der Drehgeschwindigkeit keine zusätzlichen Beiträge zur führenden und der darauffolgenden Ordnung zur oberen Schranke der GP Energie aufkommen. Des Weiteren finden wir unter Verwendung einer Variationsgleichung, die mit einem weiteren GP Funktional assoziert ist, dass es eine additive Zerlegung des GP Energiefunktionals gibt. Dies führt zu der getrennten Betrachtung von zwei verschiedenen Funktionalen. Das eine berücksichtigt den Beitrag zur Energie durch die Wirbel im Kondensat und das andere den Beitrag des Profiles des Bose-Einstein Kondesates. Diese Methode liefert im Prinzip bessere Schranken als die vorherige.

In this work we study a rapidly rotating Bose-Einstein condensate on a disc in the strong coupling (Thomas-Fermi) limit in the 2D Gross-Pitaevskii (GP) framework. We establish upper and lower bounds to the energy of the Bose Gas under Dirichlet boundary conditions. The case of Neumann boundary conditions has already been considered in [CY] and [CDY]. In the present work these considerations are extended by including an additional factor in the trial function for the upper bound to ensure Dirichlet boundary conditions. We show explicitly that this does not effect the contribution to the leading and subleading order of the GP energy in a certain regime of the rotational velocity. Moreover, by using a variational equation associated with the minimizer of an auxiliary GP functional an additive decoupling of the GP energy functional is achieved. This leads us to consider two functionals seperately, one accounting for the contribution of the vortices and the other for the density profile of the Bose-Einstein condensate. This method leads in principle to better bounds than the previous approach.

Contents

2 Introduction

2.1 Historical Overview

In 1924 S.N. Bose derived Planck's formula for the spectral distribution of the energy of photons in a cavity by using a new way of counting the possible states that differed from classical Boltzmann statistics [Bo]. Einstein [E] realized that the method of Bose could also be used in the quantum theory of an ideal gas of material particles. Moreover, he discovered in the same work that in thermal equilibrium at sufficiently low temperature (depending on the density) a macroscopic number of particles of the gas occupied the ground state. We now refer to this phenomenom as Bose-Einstein condensation (BEC), but in those years it was more a mathematical feature than an experimentally proven fact. Shortly afterwards the formalism of quantum mechanics was developed, and the statistics of particles were implemented in the symmetry properties of the many-particle wave function: Symmetric wave functions decribe bosons with integer spin and antisymmetric fermions with half-integer spin. This distinction implies that only bosons reach a common single state. Liquid Helium (^{4}He) was suggested first to be a candidate for the realization of a BE condensate, but the situation has not yet be clarified rigorously, because the strong interaction between the helium atoms has to be taken into account. Bogoliubov [Bog] analyzed systematically a weakly interacting Bose gas at $T = 0$K and derived through an ingenious approximation to the Hamiltonian (written in terms of creation and annihilation operators) the energy of the ground state and of excitations by quasiparticles.

An experimental breakthrough was achieved by Cornell and Wieman [CW] and a short time later in the same year with a different approach (a novel trap) by the Ketterle [K1] group. Both groups experimentally realized BEC, the first with rubidium and the latter with a sodium isotope and a three orders of magnitude larger number of Bose-condensed atoms. A wide variety of experimental setups followed, carried out worldwide by different scientists, and BE condensates became and still are an important area in research [K2]. Extensive theoretical studies have also been performed. For these the Gross-Pitaevskii (GP) framework has been very important. The GP equation, a nonlinear (cubic) Schrödinger equation, has the advantage to be less complex compared to the many particle Schrödinger equation. The monograph [A] and the review article [F] describe extensively the mathematical modelling of the Bose-Einstein condensate and in particular the GP theory for different traps and its connection to recent experiments. As universe of discourse many investigations have the effect of rotation on the formation of BE condensates. Indeed, these condensates respond to rotation with a wide range of phenomena, such as the appearance of quantized vortices in a particular shape and their arrangement. Experimentally, quantized vortices in BE condensates where realized in 1999 for example by the Cornell and Wiemann group [CW2]. On the other hand a mathematically rigorous proof of the asymptotical exactness of the GP description of the Bose gas with repulsive interaction was derived by Lieb, Seiringer and Yngvason [LSY], [LS1] and additionally the existence of 100% BEC [LS1], [LS2]. It was shown in [SchY] that a 2D GP functional also reproduces the energy of the many body problem with strong confinement in one direction or for highly elongated traps.

3 Mathematical Framework and Methods

3.1 Hamiltonian

As starting point we consider a Hamiltonian for N interacting particles acting on bosonic wave functions, i.e., on the symmetric part of a Hilbert space $L^2(\mathbb{R}^{3N}, d\vec{x}_1, \dots d\vec{x}_N)$,

$$H = \sum_{i=1}^{N} H_0^{(i)} + \sum_{1 \le i < j \le N} v(|\vec{x}_i - \vec{x}_j|), \tag{3.1}$$

where v is a spherically symmetric, positive two-particle interaction, which decreases faster than $|\vec{x}|^{-3}$ at infinity, and H_0 is a one particle Hamiltonian. We use units in which $\hbar = 2m = 1$, where m denotes the mass of a single particle. Since we are interested in identical bosons in a rotating trap, we consider the one-particle quantum mechanical Hamiltonian in a rotating reference frame of the form

$$H_0 = -\Delta - \vec{L} \cdot \vec{\Omega} + V(\vec{x}) \tag{3.2}$$

acting on the one-particle Hilbert space $L^2(\mathbb{R}^3, d\vec{x})$. $H_0^{(i)}$ denotes the corresponding operator for the particle i. Here $\vec{\Omega}$ denotes the rotational velocity, $V(\vec{x})$ the trapping potential, $\vec{x}$ the position operator and $\vec{L} = -i\vec{x} \times \vec{\nabla}$ its angular momentum operator. We choose $\vec{\Omega} = \Omega\vec{e}_z$ and by introducing the vector potential $\vec{A} = \frac{1}{2}\Omega(\vec{e}_z \times \vec{x})$, where $\vec{e}_z$ is the unit vector in z-direction, we are able to write (3.1) as

$$H = \sum_{i=1}^{N} \left\{ \left(\mathrm{i}\vec{\nabla}_j + \vec{A}(\vec{x}_j) \right)^2 + V(\vec{x}_j) - \frac{1}{4}\Omega^2 x_j^2 \right\} + \sum_{1 \le j < k \le N} v(|\vec{x}_j - \vec{x}_k|) \tag{3.3}$$

with the distance from the rotational axis $r \equiv |\vec{e}_z \times \vec{x}|$. In the rotating frame the vector potential $\vec{A}$ can be seen as the Coriolis part of this Hamiltonian and $-\Omega^2 r^2/4$ contributes the centrifugal force. In order to confine the condensate we require for the trapping potential $V(\vec{x}) : \mathbb{R}^3 \to \mathbb{R}$

$$V(\vec{x}) \geq \Omega^2 x^2/4 \tag{3.4}$$

as $x \equiv |\vec{x}| \to \infty$. The simplest way to fulfill this condition is to assume $V(\lambda\vec{x}) = \lambda^s V(\vec{x})$ for $\lambda > 0$ with $s > 2$.

We denote the ground state energy of (3.1) by E^{QM} and the corresponding many-particle wave function Ψ_0. The particle density of the ground state is defined by

$$\rho^{\mathrm{QM}}(\vec{x}) \equiv N \int |\Psi_0(\vec{x}, \vec{X})|^2 d\vec{x}_2 \cdots d\vec{x}_N \tag{3.5}$$

with the notation $\vec{X} \equiv (\vec{x}_2, \ldots, \vec{x}_N)$.

3.1.1 Two-Particle Scattering Length a

The scattering length a is defined by means of the zero scattering Schrödinger equation

$$(-2\Delta + v(x))\,\psi = 0 \tag{3.6}$$

with the boundary condition $\psi(\vec{x}) \to 1$ as $|\vec{x}| \to \infty$. If the potential has finite range R_0, then the solution satisfies for $|\vec{x}| > R_0$

$$\psi(\vec{x}) = 1 - \frac{a}{|\vec{x}|} \tag{3.7}$$

which defines a. We also note that if v_1 has scattering length 1, then $v_a(\vec{x}) = a^{-2}v_1(\vec{x}/a)$ has scattering length a.

3.2 Bose-Einstein Condensation

We now define the term ‘Bose-Einstein condensation’ using the notion of second quantization: First we introduce creation- and annihilation operators $a(\varphi)^*$ and $a(\varphi)$ for a one-particle state φ. Let $\langle\cdot\rangle$ denote some many-particle state. Then the expectation value for the occupation number of φ in the state $\langle\cdot\rangle$ is given by

$$N_\varphi = \langle a(\varphi)^* a(\varphi) \rangle \tag{3.8}$$

and the total particle number is given by

$$N = \sum_i \langle a(\psi_i)^* a(\psi_i) \rangle, \tag{3.9}$$

where $\{\psi_i\}$ is a complete orthonormal basis of the one-particle Hilbert space $L^2(\mathbb{R}^3, d\vec{r})$.

Definition 3.1 (Bose-Einstein Condensation in a State φ [OP])
There is a $c > 0$ such that

$$N_\varphi / N \geq c \tag{3.10}$$

for all large N.

In this sense we have macroscopic occupation of a single-particle state. If there is a unique macroscopic φ we may describe a BE condensate by this single complex-valued wave function. It is then referred to as the wave function of the condensate.

The concept of BEC can also be formulated in terms of the one-particle density matrix

$$\gamma(\vec{x}, \vec{x}') = \langle a(\vec{x})^* a(\vec{x}') \rangle. \tag{3.11}$$

Here BEC means that this density matrix has at least one eigenvalue which is $\mathcal{O}(N)$.

3.3 The Gross-Pitaevskii Framework

Theoretical investigations of dilute Bose gases at low temperatures are usually carried out within Gross-Pitaevskii theory [G], [P]. Its connection with the quantum mechanical many body problem was established rigorously in [LSY] for nonrotating gases and in [LS2] for the rotating case. In the GP theory we consider an energy functional [DGP] of complex-valued wave functions $\phi(\vec{x})$ with $\vec{x} \in \mathbb{R}^3$. It is given by

$$\mathcal{E}^{\mathrm{GP}}[\phi] = \langle \phi | H_0 | \phi \rangle + 4\pi a / N \int_{\mathbb{R}^3} |\phi(\vec{x})|^4 d\vec{x}. \tag{3.12}$$

The GP energy E^{GP} is defined as the infimum of $\mathcal{E}^{\mathrm{GP}}[\phi]$ over all ϕ under the mass constraint $\|\phi\|_2^2 = N$. This infimum is indeed a minimum, i.e., there exists a ϕ^{GP} with

$$E^{\mathrm{GP}} \equiv \inf_{\|\phi\|_2 = N} \mathcal{E}^{\mathrm{GP}}[\phi] = \mathcal{E}^{\mathrm{GP}}[\phi^{\mathrm{GP}}]. \tag{3.13}$$

The GP energy fulfills the scaling relation

$$E^{\mathrm{GP}}_{N,a} = N E^{\mathrm{GP}}_{1,Na}. \tag{3.14}$$

The minimizer ϕ^{GP} is in general not unique in the rotating case [LS1], but every minimizer satisfies a variational equation, the so called GP equation,

$$(-\mathrm{i}\vec{\nabla} + \vec{A}(\vec{x}))^2 \phi(\vec{x}) + V(\vec{x})\phi(\vec{x}) + 8\pi a |\phi(\vec{x})|^2 \phi(\vec{x}) = \mu^{\mathrm{GP}} \phi. \tag{3.15}$$

Here μ^{GP} is the chemical potential given by

$$\mu^{\mathrm{GP}} = dE^{\mathrm{GP}}(N,1)/dN = E^{\mathrm{GP}}(N,1)/N + (4\pi a/N) \int_{\mathbb{R}^3} |\phi^{\mathrm{GP}}(\vec{x})|^4 d\vec{x}. \tag{3.16}$$

The GP density is defined as

$$\rho^{\mathrm{GP}}_{N,a}(\vec{x}) \equiv |\phi^{\mathrm{GP}}(\vec{x})|^2 \tag{3.17}$$

and satisfies the scaling relation

$$\rho^{\mathrm{GP}}_{N,a}(\vec{x}) = N\rho^{\mathrm{GP}}_{1,Na}(\vec{x}). \tag{3.18}$$

We define the coupling constant g as

$$g \equiv aN. \tag{3.19}$$

Next, we define the so-called GP limit, in which the GP energy E^{GP} becomes equivalent to E^{QM}.

Definition 3.2 (The GP Limit)
As $N \to \infty$ we fix the external trapping potential V and simultaneously let the interparticle potential v be scaled with N, such that a is related to N by the condition

$$Na = g \qquad \textit{fixed.} \tag{3.20}$$

More precisely, $v(\vec{x}) = a^2 v_1(\vec{x}/a)$, where v_1 is fixed with scattering length 1, and $a = g/N$.

The mean GP density $\overline{\rho}^{\mathrm{GP}}$ is defined by

$$\overline{\rho}^{\mathrm{GP}} \equiv \frac{1}{N}\int |\rho^{\mathrm{GP}}_{N,a}(\vec{x})|^2 d\vec{x}. \tag{3.21}$$

Per definition $a \sim N^{-1}$. Inserting (3.18) into (3.21) yields $a^3\overline{\rho}^{\mathrm{GP}} \sim N^{-2}$, i.e., we consider dilute systems in the GP limit,

$$a^3\overline{\rho}^{\mathrm{GP}} \ll 1. \tag{3.22}$$

The ground state energy E^{QM} of (3.1) is a function of the confining potential V, the interaction potential v and N. As in [S1] we introduce for V and v_1 fixed with v fulfilling above scaling relations the notation $E^{\mathrm{QM}}(N,a)$. It turns out that in the GP limit the minimum of the GP functional E^{GP} correctly describes the ground state energy of a trapped, non-rotating and rotating Bose gas with repulsive two-body interaction [LSY], [LS2], whereby in the latter case Ω is fixed.

Theorem 3.1 (In the GP Limit the QM Ground State Energy is E^{GP})
If $N \to \infty$ with g fixed, then

$$\lim_{N\to\infty} \frac{E^{\mathrm{QM}}(N,a)}{N} = E^{\mathrm{GP}}(1,g). \tag{3.23}$$

For a non-rotating gas ($\vec{\Omega} = 0$) it is proven that in the GP limit the GP density ρ^{GP} converges to ρ^{QM} in the weak L^1 sense [LSY]. In a rotating system the possibility of rotational symmetry breaking and ensuring nonuniqueness of the GP minimizer has to be taken into account. Furthermore, the absolute many-body ground state is in general not the same as the bosonic ground state Ψ_0.

3.3.1 Complete BEC in the GP Limes

An important fact derived in [LS1] and [LS2] is that there is complete BEC in the GP limit. In particular, when we consider a non-rotating gas, the wave function of the condensate is the unique minimizer of the GP functional ϕ^{GP}.

Theorem 3.2 (BEC in the GP Limit, Nonrotating Case)
For each fixed g

$$\lim_{N\to\infty} \frac{1}{N}\gamma(\vec{x},\vec{x}') = \phi^{\mathrm{GP}}(\vec{x})\phi^{\mathrm{GP}}(\vec{x}') \tag{3.24}$$

converges in the sense that $\mathrm{tr}\,|\frac{1}{N}\gamma - P^{\mathrm{GP}}| \to 0$ *where* $P^{\mathrm{GP}} \equiv |\phi^{\mathrm{GP}}\rangle\langle\phi^{\mathrm{GP}}|$.

There is actually more to explain about the BE condensation of a rotating Bose gas [LS2]: Let an approximate ground state be defined as a sequence of bosonic N-particle density matrices γ_N for which $\lim_{N\to\infty} N^{-1}\operatorname{tr} H\gamma_N = E^{\mathrm{GP}}(1,g)$. We denote the reduced density matrix to γ_N as $\gamma_N^{(1)}$. This matrix is a positive trace class operator on the one-particle Hilbert space $L^2(\mathbb{R}^3)$ and in the following it is normalized to $\mathrm{Tr}\gamma_N^{(1)} = 1$. Now, by applying the Banach-Alaoglu Theorem (see for example [LL]) Lieb and Seiringer find that any sequence $\gamma_N^{(1)}$ has a subsequence that converges to a γ in weak-* topology, i.e., $\lim_{N\to\infty}\mathrm{Tr}A\gamma_N^{(1)} = \mathrm{Tr}A\gamma$ for every compact operator A. Additionaly, one can show that $\mathrm{Tr}\gamma = \lim_{N\to\infty}\mathrm{Tr}\gamma_N = 1$. As a consequence we have $\lim_{N\to\infty}\mathrm{Tr}|\gamma_N^{(1)} - \gamma| = 0$. In fact, considering only positive operators, weak-* convergence together with convergence of the trace implies convergence in trace norm.

Let us now define the set of all γ's that are limit points of one-particle density matrices of approximate ground states. That is,

$$\Gamma = \left\{\gamma : \text{there is a sequence } \gamma_N, \lim_{N\to\infty}\frac{1}{N}\operatorname{tr} H\gamma_N = E^{\mathrm{GP}}(1, g = Na), \lim_{N\to\infty}\gamma_N^{(1)} = \gamma\right\}. \tag{3.25}$$

The convergence of $\gamma_N^{(1)} \to \gamma$ can either mean weak-* convergence or norm convergence. In particular norm convergence implies that $\operatorname{tr}\gamma = 1$ for all $\gamma \in \Gamma$.

Theorem 3.3 (BEC in the GP Limit for Rotating Bose Gases)
The set γ of one-particle density matrices of approximate ground states, as defined in (3.25), has the following properties.

1. *Γ is a compact and convex subset of the set of all trace class operators.*
2. *Let $\Gamma_{ext} \subset \Gamma$ denote the set of extreme points in Γ. (An element $\gamma \in \Gamma$ is extreme if γ cannot be written as $\gamma = a\gamma_1 + (1-a)\gamma_2$ with $\gamma_{1,2} \in \Gamma$, $\gamma_1 \neq \gamma_2$ and $0 < a < 1$.) We have $\Gamma_{ext} = \{|\phi^{\mathrm{GP}}\rangle\langle\phi^{\mathrm{GP}}| : \mathcal{E}^{\mathrm{GP}}[\phi^{\mathrm{GP}}] = E^{\mathrm{GP}}(g)\}$, i.e., the extreme points in Γ are given by the rank-one projections onto* GP *minimizers*
3. *For each $\gamma \in \Gamma$, there is a positive (regular Borel) measure $d\mu_\gamma$, supported in Γ_{ext}, with $\int_{\Gamma_{ext}} d\mu_\gamma(\phi^{\mathrm{GP}}) = 1$, such that*

$$\gamma = \int_{\Gamma_{ext}} d\mu_\gamma(\phi^{\mathrm{GP}})|\phi^{\mathrm{GP}}\rangle\langle\phi^{\mathrm{GP}}|, \tag{3.26}$$

where the integral is understood in the weak sense. That is, every $\gamma \in \Gamma$ is a convex combination of rank-one projetions onto GP *minimizers.*

By assuming the trapping potential to be slightly asymmetric, a minimizer in which to condensate might be distinguished (and it is expected that Γ_{ext} then contains only one element), such that one has complete BEC.

3.4 GP Energy in 2D and its Correspondence to the Many Body Problem

We now restrict our consideration to a 2D domain. It is applicable if one considers strong confinement of the condensate in one direction. A rigorous proof of the crossover from the Hamiltonian (3.1) without rotation ($\vec{\Omega} = 0$) to the GP energy in 2D has been achieved in [SchY]. We explain their setting and the main theorem shortly: The particles are strongly confined in z-direction. We introduce the notation $(\vec{r}, z) = \vec{x} \in \mathbb{R}^3$, with $\vec{r} \in \mathbb{R}^2$ and $z \in \mathbb{R}$. Further on, we consider trapping potentials of the form

$$V_{L,h}(\vec{x}) = V_L(\vec{r}) + V_h^{\perp}(z) = \frac{1}{L^2}V(L^{-1}\vec{r}) + \frac{1}{h^2}V^{\perp}(h^{-1}z) \tag{3.27}$$

with V and $V^{\perp}$ fixed and consider h and L as parameter. Additionally, v has the parameter a. It is assumed that $V(x)$ and $V^{\perp}(z)$ are locally bounded and $V(x), V^{\perp}(z) \to \infty$ as $|x|, |z| \to \infty$. So, the ground state energy of (3.1) for $\vec{\Omega} = 0$ has the scaling property

$$E^{\mathrm{QM}}(N, h, a, L) = \frac{1}{L^2} E^{\mathrm{QM}}(N, h/L, a/L, 1). \tag{3.28}$$

In contrast to the GP limes we have considered before the ratio h/L is not fixed, but tends to zero. We denote the ground state energy of $-d^2/dz^2 + V^{\perp}(z)$ by $e^{\perp}$ and the normalized ground state wave function as $s(z)$.

We now turn to the two-dimensional GP functional defined analogously as in 3D, but we consider the domain of the wave functions to be $\mathbb{R}^2$. Additionally the coupling parameter is given by

$$g^{2\mathrm{D}} = |\ln(\overline{\rho} a_{2\mathrm{D}}^2)|^{-1}. \tag{3.29}$$

For simplicity we mean in the following $g^{2\mathrm{D}}$ but write g. We define the mean density $\overline{\rho}$ as

$$\overline{\rho}_{Ng} = \frac{1}{N} \int |\varphi_{Ng}^{\mathrm{GP}}|^4 d\vec{r} \tag{3.30}$$

where $\varphi^{\mathrm{GP}}(\vec{r})_{Ng}$ is a minimizer of the 2D GP functional. The 2D scattering length is defined by

$$a_{2\mathrm{D}} = h \exp\left(- \left(\int s(z)^4 dz \right)^{-1} h/2a \right). \tag{3.31}$$

The GP energy per particle in 2D is defined as

$$E_{2\mathrm{D}}^{\mathrm{GP}}(N, L, g)/N = \inf \left\{ \mathcal{E}_{2\mathrm{D}}^{\mathrm{GP}}[\varphi], \int |\varphi(\vec{x})|^2 d^2\vec{x} = 1 \right\} = \frac{1}{L^2} E_{2\mathrm{D}}^{\mathrm{GP}}(1, 1, Ng), \tag{3.32}$$

where the last step points out the scaling of $V^{\perp}$ and the GP minimizer.

Theorem 3.4 (From 3D to 2D, Ground State Energy)
Let $N \to \infty$ and at the same time $h/L \to 0$ and $a/h \to 0$ in such a way that $h^2 \overline{\rho} g^{2\mathrm{D}} \to 0$ (with $g^{2\mathrm{D}}$ given by (3.29)). Then

$$\lim \frac{E^{\mathrm{QM}}(N, L, h, a) - N h^{-2} e^{\perp}}{E_{2D}^{\mathrm{GP}}(N, L, g)} = 1. \tag{3.33}$$

The meaning of $h^2 \overline{\rho} g \to 0$ is that the ground state energy $h^{-2} e^{\perp}$ associated with the confining potential in the $z-$direction is much larger than the energy $\overline{\rho} g$. This is the so-called condition of strong confinement.

Remark: In our proof we consider a rapidly rotating Bose gas in 2D. We assume that (3.33) is true even for $\Omega \to \infty$ and $g^{2\mathrm{D}} \to \infty$ (possibly with some additional constraint on diluteness), although this still remains to be proved rigorously.

3.5 A Rapidly Rotating Bose Gas on a Disc

Now, we consider a 2D GP functional of wave functions on a disc of finite radius and center at the origin. We write the 2D coupling parameter as $g^{2\mathrm{D}} = 1/\varepsilon^2$ and consider a regime of 'rapid rotation' by which we mean

$$|\log \varepsilon| \ll \Omega \ll \frac{1}{\varepsilon^2 |\log \varepsilon|} \tag{3.34}$$

with $\varepsilon \to 0$. We assume the rotational axis to be perpendicular to the disc and to be located at its center. Moreover, the confining potential $V(\vec{r})$ is assumed to be zero inside a finite radius R and infinite beyond, which in the picture of homogeneous potentials $V(\vec{r}) \sim (r/R)^s$ can be seen as the case where $s \to \infty$. A feature of such a potential is that the condensate is strongly confined and cannot be blown apart by any centrifugal force. By scaling, we may choose the length unit so that $R = 1$. We remark that the limit $s = \infty$ naturally leads to the (zero) Dirichlet boundary condition. We denote the position in the disc by $\vec{r} = (x, y)$. In this setting the GP energy functional in the non-inertial rotating frame can be written as

$$\mathcal{E}^{\mathrm{GP}}[\Psi] = \int_{\mathcal{B}_1} d\vec{r} \left\{ \left| \left(\vec{\nabla} - i\vec{A} \right) \Psi \right|^2 - \frac{\Omega^2 r^2 |\Psi|^2}{4} + \frac{|\Psi|^4}{\varepsilon^2} \right\}, \tag{3.35}$$

where $\mathcal{B}_1$ denotes a ball (disc) of radius 1. It is considered on the domain

$$\mathcal{D}^{\mathrm{GP}} = H_0^1(\mathcal{B}_1) \tag{3.36}$$

whereby $H_0^1(\mathcal{B}_1)$ is defined as the Sobolev space $H^1 = \{\Psi : \|\Psi\|_2^2 + \|\vec{\nabla}\Psi\|_2^2 < \infty\}$ with zero Dirichlet boundary condition, i.e., Ψ is zero on $\partial\mathcal{B}_1$. A Sobolev inequality implies that $H_0^1(\mathcal{B}_1)$ is contained in $L^4(\mathcal{B}_1)$, such that the integration of $|\Psi|^4$ is well defined. We denote a minimizer of (3.35) by Ψ^{GP}. Hence, the GP ground state energy is

$$E^{\mathrm{GP}} \equiv \inf_{\|\Psi\|_2=1} \mathcal{E}^{\mathrm{GP}}[\Psi] = \mathcal{E}^{\mathrm{GP}}[\Psi^{\mathrm{GP}}]. \tag{3.37}$$

We now introduce the abbreviation

$$\omega \equiv \varepsilon\Omega. \tag{3.38}$$

If ω is fixed the centrifugal and the interaction term in (3.35) are $\mathcal{O}(1/\varepsilon^2)$. Since $A \sim \Omega$ the kinetic energy term in (3.35) might also be of the same order. But it emerges that a complex phase in this term compensates a part of the energy originating from $\vec{A}$ [CY].

As proved in [CDY] the asymptotics of E^{GP} with Neumann boundary conditions in the limit $\varepsilon \to 0$ is given to leading order by the infimum of the so called Thomas-Fermi (TF) functional,

$$\mathcal{E}^{\mathrm{TF}}[\rho] = \frac{1}{\varepsilon^2} \int_{\mathcal{B}_1} d\vec{r} \left\{ \rho^2 - \frac{\omega^2 r^2 \rho}{4} \right\}. \tag{3.39}$$

It is defined over the domain

$$\mathcal{D}^{\mathrm{TF}} = \{\rho : \rho \in L_2(|\vec{r}| \leq 1) \text{ and } \rho \geq 0\}. \tag{3.40}$$

The TF ground state energy is defined by

$$E^{\mathrm{TF}} \equiv \min_{\|\rho\|_1=1, \rho \geq 0} \mathcal{E}^{\mathrm{TF}}[\rho] = \mathcal{E}^{\mathrm{TF}}[\rho^{\mathrm{TF}}] \tag{3.41}$$

where ρ^{TF}, the so-called TF density, is defined as the minimizing density. As ω becomes larger than $\omega_h \equiv 4/\sqrt{\pi}$ the TF density has a hole at the center of the disc. We remark that we have collected other useful properties and explicit formulas for the TF energy and density in the Appendix A.

We will later decouple the GP energy functional (3.35) additively into two seperate functionals. One of them is as (3.35) a GP type energy functional, but without the vector potential term $\vec{A}$. It is defined as

$$\hat{\mathcal{E}}^{\mathrm{GP}}[\varphi] \equiv \int_{\mathcal{B}_1} \left\{ |\vec{\nabla}\varphi|^2 - \frac{1}{4} r^2 \Omega^2 |\varphi|^2 + \frac{1}{\varepsilon^2} |\varphi|^4 \right\} \tag{3.42}$$

on $\mathcal{D}^{\mathrm{GP}}$. It accounts for the energy of the profile of the rotating Bose-Einstein condensate. As we will show, it is possible to restrict the Dirichlet boundary condition to the minimizer of (3.42), while the domain of the remaining functional does not include this constraint. The variational equation satisfied by the minimizer of (3.42), denoted by g, is the GP equation [LSY],

$$-\Delta g - \frac{1}{4} r^2 \Omega^2 g + \frac{2}{\varepsilon^2} g^3 = \hat{\mu}^{\mathrm{GP}} g, \tag{3.43}$$

with a Lagrangian multiplier $\hat{\mu}^{\mathrm{GP}}$ to take the mass constraint $\|\varphi\|_2 = 1$ into account. Moreover, the minimizer of (3.42) is unique up to a constant phase, real-valued and strictly positive for $|\vec{r}| < 1$ (see [LSY]). As a consequence of the uniqueness, g is rotationally symmetric, i.e., $g(\vec{r}) = g(r)$. The GP energy of the profile is defined by the infimum of (3.42), i.e.,

$$\hat{E}^{\mathrm{GP}} \equiv \inf_{\varphi \in \mathcal{D}_{\mathrm{GP}}; \|\varphi\|_2 = 1} \hat{\mathcal{E}}^{\mathrm{GP}}[\varphi] \tag{3.44}$$

which is indeed a minimum, i.e., there is a (unique, positive) $g \in \mathcal{D}^{\mathrm{GP}}$ such that $\hat{E}^{\mathrm{GP}} = \hat{\mathcal{E}}^{\mathrm{GP}}[g]$. The chemical potential $\hat{\mu}^{\mathrm{GP}}$ follows from (3.43) by multiplying with g and integrating. As result we have

$$\hat{E}^{\mathrm{GP}} + \varepsilon^{-2} \|g\|^4_{L_4(\mathcal{B}_1)} = \hat{\mu}^{\mathrm{GP}}. \tag{3.45}$$

3.5.1 Spherical Symmetry and the Emergence of Vortices

Let us compare a 2D BE condensate with sufficiently small ε on a disc to a condensate in a trap which is polynomally bounded at infinity.

A condensate confined in a trap that is polynomally bounded at infinity responds to rotation by breaking the spherical symmetry [S2]: Let us change from cartesian $\vec{r} = (x, y)$ to polar coordinates denoted by (r, φ) and rewrite the angular momentum operator as $L = -\mathrm{i}\partial/\partial\varphi$. It is proved, for all $0 < \Omega < \Omega_c$, with $\Omega_c \in \mathbb{R}^+ \cup \infty$, there is an ε_Ω such that $\varepsilon \leq \varepsilon_\Omega$ implies that no ground state of the 2D GP functional is an eigenfunction of L, i.e., $L\phi^{\mathrm{GP}} = n\phi^{\mathrm{GP}}$, and additionally one finds the following:

Theorem 3.5 (Symmetry Breaking)
Let $0 < \Omega < \Omega_c$ and $\varepsilon \leq \varepsilon_\Omega$, and let ϕ^{GP} be a minimizer of the 2D GP *functional. Then $|\phi^{\mathrm{GP}}|$ is not a radial function.*

Traps with infinite high walls, i.e., if $s = \infty$, (and fixed radius of the trap) imply that for fixed Ω, provided ε is sufficiently small, the ground state is a unique, strictly positive and radial function [CDY]. Indeed, vortices, i.e., isolated zeros of the minimizer, appear when it becomes energetically favorable. This is the case when the energy contribution of a vortex is smaller than the modulus of the energy originating from the rotation of the condensate [A], [CDY]. On this phenomenon, two statements were proved in [CDY] with Neumann boundary condition by Correggi, Rindler-Daller and Yngvason.

Theorem 3.6 (Instability for Higher Vorticity)
Let $\Psi_n(\vec{r}), n \geq 2$, be the unique minimizer of $\mathcal{E}^{\mathrm{GP}}[\Psi]$ on the subspace of functions with angular momentum n, i.e., on $\{\Psi \in \mathcal{D}^{\mathrm{GP}} | L\Psi = n\Psi\}$. For any $\Omega \leq 1/(\sqrt{\pi}\varepsilon)$, Ψ is unstable, i.e., it is not a local minimizer of $\mathcal{E}^{\mathrm{GP}}[\Psi]$.

As a consequence they show symmetry breaking of the minimizer Ψ^{GP}.

Theorem 3.7 (Symmetry Breaking in the Ground State)
For ε sufficiently small, no minimizer of $\mathcal{E}^{\mathrm{GP}}[\Psi]$ is an eigenfunction of the angular momentum, if

$$6|\log \varepsilon| + 3 < \Omega \leq \frac{c}{\varepsilon} \tag{3.46}$$

for any constant $c \in \mathbb{R}^+$.

The reader may ask, how do vortices arrange themselves in the condensate? A statement about a energetically favorable distribution of vorticity for $|\log \varepsilon| \ll \Omega \lesssim \varepsilon^{-1}$ can be found in [CY]. As a consequence for our proof of the upper bounds to E^{GP} we consider a regular lattice as distribution of vortices.

4 Main Results

In this work we investigate the effect of the Dirichlet boundary condition on the GP energy for the parameter range $1 \ll \Omega \ll 1/\varepsilon^2$. To formulate our theorems we introduce the symbol C that denotes an adequate constant which value may change from line to line. Now, the main results are:

Theorem 4.1 (Energy Upper Bound)
As $\varepsilon \to 0$, we have for $1 \ll \Omega \lesssim 1/\varepsilon$

$$E^{\mathrm{GP}} \leq E^{\mathrm{TF}} + \frac{C}{\varepsilon} + \frac{\Omega}{2}|\log(\varepsilon^2\Omega)|(1+o(1)) \tag{4.1}$$

and for $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$

$$E^{\mathrm{GP}} \leq E^{\mathrm{TF}} + \frac{\Omega}{2}|\log\varepsilon|(1+o(1)) + C\varepsilon^{1/2}\Omega^{3/2}. \tag{4.2}$$

By this upper bounds we find together with the lower bounds in [CY] the following.

Theorem 4.2 (Energy Asymptotics)
For ε sufficiently small, and if Ω satisfies $|\log\varepsilon| \ll \Omega \lesssim (\varepsilon|\log(\varepsilon^2\Omega)|)^{-1}$, we have

$$E^{\mathrm{GP}} = E^{\mathrm{TF}}(1+o(1)). \tag{4.3}$$

If $(\varepsilon|\log(\varepsilon^2\Omega)|)^{-1} \ll \Omega \lesssim \varepsilon^{-1}$, then

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \frac{\Omega}{2}|\log(\varepsilon^2\Omega)|(1+o(1)) \tag{4.4}$$

whereas if $\varepsilon^{-1} \lesssim \Omega \ll \varepsilon^{-1}|\log\varepsilon|^2$,

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \frac{\Omega}{2}|\log\varepsilon|(1+o(1)). \tag{4.5}$$

For $\varepsilon^{-1}|\log\varepsilon|^2 \lesssim \Omega \ll (\varepsilon^2|\log\varepsilon|)^{-1}$ we have

$$E^{\mathrm{GP}} = E^{\mathrm{TF}}(1+o(1)). \tag{4.6}$$

As an important ingredient to our second proof of the energy asymptotics we find an additive decoupling of the Gross-Pitaevskii functional (3.35).

Theorem 4.3 (Energy Decoupling)
For any $\psi \in \mathcal{D}^{\mathrm{GP}}$ so that $\|\psi\|_2 = 1$, define $\psi(\vec{r}) \equiv g(r)\cdot u(\vec{r})$ where g denotes the positive minimizer of (3.42) with Dirichlet boundary condition, and $u(\vec{r})$ is complex-valued. We have the following energy decoupling

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \hat{E}^{\mathrm{GP}} + \mathcal{F}_g[u] \tag{4.7}$$

where

$$\mathcal{F}_g[u] \equiv \int_{\mathcal{B}_1}\left\{\frac{1}{\varepsilon^2}|g|^4(1-|u|^2)^2 + |\vec{\nabla}_A u|^2|g|^2\right\} \tag{4.8}$$

with the notation $\vec{\nabla}_A \equiv \vec{\nabla} - i\vec{A}$.

This decoupling enables us to find refined energy asymptotics.

Theorem 4.4 (Refined Energy Asymptotics)
For ε sufficiently small, and if Ω satisfies $|\log\varepsilon| \ll \Omega \lesssim 1/\varepsilon$ we have

$$E^{\mathrm{GP}} = \hat{E}^{\mathrm{GP}} + \frac{\Omega}{2}|\log(\varepsilon^2\Omega)|(1+o(1)), \tag{4.9}$$

and for $1/\varepsilon \lesssim \Omega \ll (\varepsilon^2|\log\varepsilon|)^{-1}$

$$E^{\mathrm{GP}} = \hat{E}^{\mathrm{GP}} + \frac{\Omega}{2}|\log\varepsilon|(1+o(1)). \tag{4.10}$$

We remark that $\hat{E}^{\mathrm{GP}}$ is computable, since its minimizer solves an ordinary second order differential equation. Starting from (4.7) we reproduce the upper bounds (4.1) and (4.2), and also the energy asymptotics (4.3), (4.4), (4.5) and (4.6).

5 Energy Asymptotics with Dirichlet Boundary Condition

It is our aim to examine the effect of the Dirichlet boundary condition. In particular we will prove in this section that for sufficiently rapid rotation it has no effect on the first and second order contribution to the asymptotic expansion of E^{GP}.

5.1 Energy Upper Bound

We start with the proof of Theorem 4.1.

Proof: Imposing the Dirichlet boundary condition to the domain of the GP functional (3.35) means requiring the wave function to be zero at the boundary of the disc $\partial\mathcal{B}_1$. To fulfill this requirement we extend the trial function of [CY] (equation 4.1) by a cut-off function which is zero at the boundary. Hence, we choose as ansatz a wave function of the form

$$\Psi(\vec{r}) \equiv c\sqrt{\rho(\vec{r})}\xi(\vec{r})g(\vec{r})d(\vec{r}). \tag{5.1}$$

Here c denotes the normalization constant, ρ a regularization of the density $\rho^{\mathrm{TF}}(\vec{r})$, $\xi(\vec{r})$ a function vanishing linearly at each vortex, i.e., at the singularities of the phase function $g(\vec{r})$, and $d(\vec{r})$ is a function that takes care of the Dirichlet boundary condition. The latter is defined by

$$d(\vec{r}) \equiv \begin{cases} 1 & \vec{r} \in \mathcal{B}_1\backslash\mathcal{I} \\ (1/\delta)\,(1-r)) & \vec{r} \in \mathcal{I} \end{cases} \tag{5.2}$$

with a parameter $1 \gg \delta > 0$ and $\mathcal{I}$ is given by

$$\mathcal{I} \equiv \{r | 1-\delta \leq r \leq 1\}.$$

The area of the annulus $\mathcal{I}$ is

$$A = 2\pi\delta\left(1 - \frac{\delta}{2}\right). \tag{5.3}$$

As in [CY] we decompose the disc $\mathcal{B}_1$ into cells $\mathcal{Q}^i$ whose centers $\vec{r}_i \in \mathcal{B}_1$ are arranged in a regular lattice denoted by $\mathcal{L}$. The lattice constant l is choosen so that each cells area is

$$|Q^i| = \frac{2\pi}{\Omega}, \tag{5.4}$$

i.e.,

$$l = C\Omega^{-1/2} \tag{5.5}$$

and the total number of lattice points inside $\mathcal{B}_1$ is given by

$$\mathcal{N} = \frac{\Omega}{2}(1 + \mathcal{O}(\Omega^{-1/2})). \tag{5.6}$$

The number of lattice points in $\mathcal{I}$ is given by

$$\mathcal{N}_{\mathcal{I}} = \mathcal{O}(\Omega\delta). \tag{5.7}$$

If $\omega = \varepsilon\Omega$ is large the area of $\operatorname{supp}\rho^{\mathrm{TF}}$ is $\mathcal{O}(\omega)^{-1}$ and the number of lattice points on the support of ρ^{TF} is of the order

$$\mathcal{N}' = \mathcal{O}\left(\frac{1}{\varepsilon}\right). \tag{5.8}$$

Since $\mathbb{C} \simeq \mathbb{R}^2$ the position vector $\vec{r} = (x, y) \in \mathbb{R}^2$ can be written as a complex number $\zeta = x + iy \in \mathbb{C}$. So, we define the phase factor g as

$$g(\vec{r}) \equiv \prod_{\zeta_i \in \mathcal{L}} \frac{\zeta - \zeta_i}{|\zeta - \zeta_i|}. \tag{5.9}$$

To get rid of the singularities of the phase factor g at the lattice points, we define the function

$$\xi(\vec{r}) \equiv \begin{cases} 1 & \text{if} \quad |\zeta - \zeta_i| > t \quad \text{for all } i \\ t^{-1}|\zeta - \zeta_i| & \text{if} \quad |\zeta - \zeta_i| \leq t \end{cases} \tag{5.10}$$

where t is a variational parameter, which fulfills $\min\{\varepsilon, (\varepsilon/\Omega)^{1/2}\} \leq t \ll \Omega^{-1/2}$ and will be fixed later. To justify the estimates on t we refer to the heuristics in [CY]. The function $\xi(\vec{r})$ generates discs $\mathcal{B}_t^i$ of radius t with center at the lattice points $\vec{r}_i \in \mathcal{L}$, where ξ vanishes, but is equal 1 in the complement to those discs.

In the case where $\Omega \leq \omega_h/\varepsilon$, we set the density ρ to be equal to the TF density ρ^{TF}. If we consider $\omega > \omega_h$ the density ρ^{TF} vanishes inside a hole of radius $R_h = 1 - C\omega^{-1}$, but is abruptly increasing in r for $R_h < r$ (see (A.4)). Hence, the kinetic energy originating from the term $\sqrt{\rho^{\mathrm{TF}}}$ is infinite. Therefore it is necessary to regularize ρ^{TF} near the boundary of the hole. But it is also important that the density remains similar to ρ^{TF} . Thus, if $\omega > \omega_h$ we define as in [CY], equation 4.9,

$$\rho(r) \equiv \begin{cases} 0 & \text{if } r \leq R_h \\ \rho^{\mathrm{TF}}(R_h + \Omega^{-1})\Omega^2(r - R_h)^2 & \text{if } R_h \leq r \leq R_h + \Omega^{-1} \\ \rho^{\mathrm{TF}}(r) & \text{otherwise} \end{cases} \tag{5.11}$$

Notice that the only difference to ρ^{TF} is that it is equal to $\rho^{\mathrm{TF}}(R_h + \Omega^{-1})\Omega^2(r - R_h)^2$ inside $R_h \leq r \leq R_h + \Omega^{-1}$, i.e., from the radius of the hole R_h the regularized density increases quadratically in an annulus of thickness $\sim \Omega^{-1}$ with increasing distance from the hole. This construction guarantees a finite kinetic energy. By (A.4) we find that

$$\rho^{\mathrm{TF}}(R_h + \Omega^{-1}) = \mathcal{O}(\varepsilon^2\Omega) \tag{5.12}$$

and so one has

$$\rho(r) = \rho^{\mathrm{TF}}(r) + \mathcal{O}(\varepsilon^2\Omega) \tag{5.13}$$

inside $\mathcal{B}_1$. Note that by assumption $\varepsilon^2\Omega = o(1)$.

Now, we calculate some useful estimates. Both functions, d^2 and ξ^2, are smaller than or equal to 1. Thus, we have

$$d^2\xi^2 \geq 1 - (1 - \xi^2) - (1 - d^2). \tag{5.14}$$

By the explicit formula for the TF density (A.2) one gets $\rho^{\mathrm{TF}} \leq C(\omega + 1)$. Also recall that the number of lattice points in support of ρ^{TF} is (5.8). Combining these facts together with (5.14), (5.13) we find

$$\int_{\mathcal{B}_1} \rho\xi^2 d^2 \geq \int_{\mathcal{B}_1} \rho - \int_{\cup_i \mathcal{B}_i^t} \rho(1 - \xi^2) - \int_{\mathcal{I}} \rho(1 - d^2) \geq 1 - C(\Omega t^2) - C(\omega + 1)\delta. \tag{5.15}$$

Thus, by the normalization condition $\|\Psi\|_2 = 1$ we get for the constant the estimate

$$c^2 \leq 1 + C\Omega t^2 + C(\omega + 1)\delta. \tag{5.16}$$

Since ρ, ξ, d are real-valued functions, $d \leq 1$ and g is a phase factor, Cauchy-Schwarz inequality yields

$$\begin{aligned} \left|\left(\vec{\nabla} - i\vec{A}\right)(\sqrt{\rho}\xi g d)\right|^2 &= |\vec{\nabla}(\sqrt{\rho}\xi d)g + i(g\xi\sqrt{\rho}d\vec{\nabla}\phi - \vec{A}d\sqrt{\rho}\xi g)|^2 = \\ &= |\vec{\nabla}(\sqrt{\rho}\xi d)|^2 + |d\xi\sqrt{\rho}(\vec{\nabla}\phi - \vec{A})|^2 \leq \\ &\leq 3\left(|\vec{\nabla}(\xi)|^2\rho + |\vec{\nabla}(\sqrt{\rho})|^2\xi^2 + |\vec{\nabla}(d)|^2\rho\xi^2\right) + |\xi\sqrt{\rho}(\vec{\nabla} - i\vec{A})g|^2. \end{aligned} \tag{5.17}$$

Integration of (5.17) gives the contribution to the kinetic energy. The corresponding estimate for the first term on the right hand side in (5.17) is the same as in [CY]

$$\|\sqrt{\rho}\vec{\nabla}(\xi)\|_2^2 \leq C\Omega. \tag{5.18}$$

Also for the second term we may use the corresponding estimate in [CY], which is given by

$$||\xi\nabla(\sqrt{\rho})||_2^2 \leq \int_{\mathcal{B}_1} |\xi|^2 \frac{1}{4}\frac{|\nabla\rho|^2}{\rho} \leq C\Omega(\varepsilon^2\Omega) + C(\varepsilon^2\Omega)\Omega|\log\varepsilon|. \tag{5.19}$$

Since $(\vec{\nabla}(d))^2 = 1/\delta^2$ for all $r \in \mathcal{I}$ and zero inside $\mathcal{B}_{1-\delta}$ we find together with $\rho \leq \rho^{\mathrm{TF}}(1) \leq C(\omega+1)$ the estimate

$$\|\vec{\nabla}(d)\sqrt{(\rho)}\xi\|_2^2 \leq \int_{\mathcal{B}_1} |\vec{\nabla}(d)|^2\rho = \frac{1}{\delta^2}\int_{\mathcal{I}}\rho \simeq \frac{1}{\delta^2}\int_{1-\delta}^1 r\rho \leq \frac{C}{\delta}(\omega+1). \tag{5.20}$$

Putting the above estimates together we obtain for the difference between the GP and the TF functional the bound

$$\begin{aligned}\mathcal{E}^{\mathrm{GP}}[\Psi] - \mathcal{E}^{\mathrm{TF}}[|\Psi|^2] = c^2 \int_{\mathcal{B}_1} | \left(\vec{\nabla} - i\vec{A}\right)(\sqrt{\rho}\xi g d)|^2 \\ \leq \left(1 + C\Omega t^2 + C(\omega+1)\delta\right)\int_{\mathcal{B}_1} \rho\xi^2 | \left(\vec{\nabla} - i\vec{A}\right) g|^2 + \\ + \left(1 + C\Omega t^2 + C(\omega+1)\delta\right)\left(C\Omega + C(\varepsilon^2\Omega)\Omega|\log\varepsilon| + \frac{C(\varepsilon\Omega+1)}{\delta}\right).\end{aligned} \tag{5.21}$$

We now claim an upper bound for the kinetic energy of the vortices.

Proposition 5.1 (Vortex Kinetic Energy)
If $\varepsilon \to 0$ and $1 \ll \Omega \ll 1/\varepsilon^2$, then

$$\int_{\mathcal{B}_1} \rho\xi^2 | \left(\vec{\nabla} - i\vec{A}\right) g|^2 \leq \frac{1}{2}\Omega|\log(t^2\Omega)| + C\Omega + C\Omega(\varepsilon^2\Omega)^{1/2}|\log(t^2\Omega)|. \tag{5.22}$$

Proof: We apply an upper bound to the kinetic energy term proved in [CY], where they use an electrostatic analogy. For completeness we repeat shortly their arguments: First we recall the definition of $g(\vec{r})$, which can be written as a phase factor,

$$g(\vec{r}) = \prod_i \frac{\zeta - \zeta_i}{|\zeta - \zeta_i|} = \exp\left(\mathrm{i}\phi(\vec{r})\right) \tag{5.23}$$

with the phase function defined by

$$\phi(\vec{r}) = \sum_i \arg(\zeta - \zeta_i). \tag{5.24}$$

The phase function is a harmonic function. Its conjugate harmonic function is

$$\chi(\vec{r}) = \sum_i \log|\vec{r} - \vec{r}_i|. \tag{5.25}$$

Thus, we find the holomorphic function

$$\chi + i\phi = \sum_i \log(\zeta - \zeta_i) \equiv \tilde{\phi}. \tag{5.26}$$

Now, by the fact that

$$|(\vec{\nabla} - i\vec{A})g|^2 = |(\vec{\nabla}\phi - \vec{A})|^2 \tag{5.27}$$

and by the Cauchy-Riemann equations we obtain the identity

$$|\vec{\nabla}\phi - \vec{A}|^2 = |\vec{\nabla}\tilde{\phi} - A\vec{e}_r|^2. \tag{5.28}$$

We now define the 'electric field'

$$\vec{E}(\vec{r}) \equiv \vec{\nabla}\tilde{\phi}(\vec{r}) - A(r)\vec{e}_r. \tag{5.29}$$

In [CY] the first term is interpreted as the electric field generated by fixed point charges located at the positions of the vortices. The second term $A(r)\vec{e}_r$ is considered as the field originating from a uniform charge density with the magnitude $\Omega/2\pi = |\mathcal{Q}^i|^{-1}$. A variable transformation maps the lattice of cells onto a lattice with side length $\mathcal{O}(1)$. A static electric field is determined by a potential, originating from a charge distribution. Thus they consider the multipole expansion of this potential, which simplifies as a consequence of the neutrality of the charge distribution and the symmetry of the unit cell. Then, each other cell $\mathcal{Q}^i$ is generated by translations and scaling starting from the first cell. This implies an estimate of the electric field in a cell generated by the other cells and together with the simple bound $\vec{E}_i(\vec{r}) \leq |\vec{r} - \vec{r}_i|^{-1}$ for any $\vec{r} \in \mathcal{Q}^i$,

$$|\vec{E}(\vec{r})|^2 \leq |\vec{E}_i(\vec{r})|^2 + const.(\Omega^{1/2}|\vec{r} - \vec{r}_i|^{-1} + \Omega). \tag{5.30}$$

A short calculation then gives

$$\int_{\mathcal{B}_1} d\vec{r}\rho^{\mathrm{TF}}(\vec{r})\xi(\vec{r})^2|\vec{E}(\vec{r})|^2 \leq \left(1 + O((t^2\Omega)^{1/2})\right)\sum_i \sup_{\vec{r}\in Q^i} \rho^{\mathrm{TF}}(\vec{r})\left(\pi|\log(t^2\Omega)| + O(1)\right). \tag{5.31}$$

It then remains to estimate the Riemann approximation error.

$$\mathcal{R} \equiv |Q^0|\sum_i \sup_{\vec{r}\in Q^i} \rho^{\mathrm{TF}}(\vec{r}) - \int_{\mathcal{B}_1} d\vec{r}\rho^{\mathrm{TF}}(\vec{r}) \leq |Q^0|\sum_i \left\{\sup_{\vec{r}\in Q^i} \rho^{\mathrm{TF}}(\vec{r}) - \inf_{\vec{r}\in Q^i} \rho^{\mathrm{TF}}(\vec{r})\right\} \tag{5.32}$$

By noting that $\|d\rho^{\mathrm{TF}}/dr\|_\infty \leq C(\varepsilon\Omega)^2$ and the number of cells Q^i that intersect ρ^{TF} is bounded by $C\varepsilon^{-1}(1 + \Omega^{-1/2})$ and the fact that the lattice constant is $l = \Omega^{-1/2}$ it follows

$$\mathcal{R} \leq C\Omega^{-1} \cdot \Omega^{-1/2}(\varepsilon\Omega)^2 \cdot \varepsilon^{-1}(1 + \Omega^{-1/2}). \tag{5.33}$$

Thus, (5.31) is bounded by

$$\begin{aligned}&\left(1 + \mathcal{O}((t^2\Omega)^{1/2})\right)(1 + \mathcal{R})\left(\pi|\log(t^2\Omega)| + \mathcal{O}(1)\right) \leq \\ &\qquad \leq \frac{1}{2}\Omega|\log(t^2\Omega)| + C\Omega + C\Omega(\varepsilon^2\Omega)^{1/2}|\log(t^2\Omega)|.\end{aligned} \tag{5.34}$$

□

Inserting (5.22) and (5.16) in (5.21) yields

$$\begin{aligned}&\mathcal{E}^{\mathrm{GP}}[\Psi] - \mathcal{E}^{\mathrm{TF}}[|\Psi|^2] \leq \left(1 + C\Omega t^2 + C(\omega + 1)\delta\right) \\ &\qquad \left(\frac{1}{2}\Omega|\log(t^2\Omega)| + C\Omega(\varepsilon^2\Omega)^{1/2}|\log(t^2\Omega)| + C\Omega + C(\varepsilon^2\Omega)\Omega|\log\varepsilon| + \frac{C(\varepsilon\Omega + 1)}{\delta}\right).\end{aligned} \tag{5.35}$$

Now, we calculate an estimate for the difference between $\mathcal{E}^{\mathrm{TF}}[|\Psi|^2]$ and $E^{\mathrm{TF}} = \mathcal{E}^{\mathrm{TF}}[\rho^{\mathrm{TF}}]$. We consider both terms of this functional separately. Since $\xi^2\rho \leq \rho^{\mathrm{TF}}$ and by inserting the normalization constant (5.16) it follows that the first term on the right hand side of (3.39) can be estimated by

$$\begin{aligned} \varepsilon^{-2}\int_{\mathcal{B}_1} d\vec{r}|\Psi|^4 = \varepsilon^{-2}\int_{\mathcal{B}_1} d\vec{r}(c^2\xi^2 d^2\rho)^2 \leq \\ \leq \frac{1+Ct^2\Omega + C(\omega+1)\delta}{\varepsilon^2}\int_{\mathcal{B}_1} d\vec{r}(\rho^{\mathrm{TF}})^2 = \varepsilon^{-2}\int_{\mathcal{B}_1} d\vec{r}(\rho^{\mathrm{TF}})^2 + \text{remainder}. \end{aligned} \tag{5.36}$$

By $\rho^{\mathrm{TF}} \leq C(\varepsilon\Omega+1)$ and $\int \rho^{\mathrm{TF}} = 1$ we obtain as upper bound to the second term in (5.36)

$$\frac{Ct^2\Omega + C(\omega+1)\delta}{\varepsilon^2}\int_{\mathcal{B}_1} d\vec{r}(\rho^{\mathrm{TF}})^2 \leq C\left(\frac{t^2\Omega}{\varepsilon^2}(\varepsilon\Omega+1)\right) + C\left((\varepsilon\Omega+1)^2\frac{\delta}{\varepsilon^2}\right). \tag{5.37}$$

This remainder has to be small compared to $\Omega|\log(t^2\Omega)|$. By a conversion of the second term in (3.39) we find by partial integration and using the normalization of Ψ the following identity.

$$-\frac{\Omega^2}{4}\int_{\mathcal{B}_1} d\vec{r}r^2|\Psi|^2 = -\frac{\pi\Omega^2}{2} + \pi\Omega^2\int_0^1 drr\Phi(r) \tag{5.38}$$

with

$$\Phi(r) = \int_0^r dr'r'|\Psi(r')|^2. \tag{5.39}$$

Analogously, we obtain

$$-\frac{\Omega^2}{4}\int_{\mathcal{B}_1} d\vec{r}r^2\rho^{\mathrm{TF}} = -\frac{\pi\Omega^2}{2} + \pi\Omega^2\int_0^1 drr\Phi^{\mathrm{TF}}(r) \tag{5.40}$$

with

$$\Phi^{\mathrm{TF}}(r) = \int_0^r dr'r'\rho^{\mathrm{TF}}(r'). \tag{5.41}$$

By the definition of the regularized density (5.11) and using the normalization constant (5.16) we find

$$\Phi(r) \leq \Phi^{\mathrm{TF}}(r) + Ct^2\Omega + C(\omega+1)\delta. \tag{5.42}$$

The area of support of Φ and Φ^{TF} is smaller than $C(\varepsilon\Omega+1)^{-1}$, which yields

$$\Omega^2\int_0^1 dr\left\{\Phi^{\mathrm{TF}}(r) - \Phi(r)\right\} \leq C\Omega^2\cdot(t^2\Omega(\varepsilon\Omega+1)^{-1} + \delta). \tag{5.43}$$

Putting together (5.35), (5.37) and (5.43), one gets for the difference between GP energy and TF energy

$$\begin{aligned} E^{\mathrm{GP}} - E^{\mathrm{TF}} \leq \big(1 + C\Omega t^2 + C(\omega+1)\delta\big) \\ \left(\frac{1}{2}\Omega|\log(t^2\Omega)| + C\Omega(\varepsilon^2\Omega)^{1/2}|\log(t^2\Omega)| + C\Omega + C(\varepsilon^2\Omega)\Omega|\log\varepsilon| + \frac{C(\varepsilon\Omega+1)}{\delta}\right) + \\ + C\Omega^2\cdot(t^2\Omega(\varepsilon\Omega+1)^{-1} + \delta) + C\left(\frac{t^2\Omega}{\varepsilon^2}(\varepsilon\Omega+1)\right) + C\left((\varepsilon\Omega+1)^2\frac{\delta}{\varepsilon^2}\right). \end{aligned} \tag{5.44}$$

It now remains to determine the parameters t and δ by minimizing the right hand side of (5.44). We distinguish two different regimes of rotation. At first we consider the regime $1 \ll \Omega \lesssim 1/\varepsilon$ where we choose $t = \varepsilon$, which implies that $C\Omega(\varepsilon\Omega)^2 \leq C\Omega$ (see (5.43)), because in this regime $\varepsilon\Omega$ is bounded. By this choice also the contribution due to t of the remainder (5.37) is small compared to $\Omega|\log(t^2\Omega)|$.

Minimizing in terms of δ yields $\delta \simeq \varepsilon$. This implies that (5.43) is $\mathcal{O}(\Omega)$, but the second term in (5.37) partly exceeds $\Omega|\log(t^2\Omega)|$. Thus we have proved the estimate

$$E^{\mathrm{GP}} \leq E^{\mathrm{TF}} + \frac{1}{2}\Omega|\log(\varepsilon^2\Omega)| + C\Omega + \frac{C}{\delta}(1 + C\Omega\varepsilon) \leq E^{\mathrm{TF}} + \frac{C}{\varepsilon} + \frac{1}{2}\Omega|\log(\varepsilon^2\Omega)| + C\Omega. \tag{5.45}$$

Next, we consider the regime $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$. Here we choose $t^2 = \varepsilon/\Omega$, which as above implies the smallness of the remainder (5.37) and the abberation originating from the centrifugal contribution (5.43) compared to $\frac{1}{2}\Omega|\log\varepsilon|$. Analogously, minimizing (5.44) in terms of δ yields $\delta \simeq \sqrt{\varepsilon/\Omega}$. Thus, the upper bound to the Gross-Pitaevskii energy in this regime is

$$E^{\mathrm{GP}} \leq E^{\mathrm{TF}} + \frac{1}{2}\Omega|\log\varepsilon| + C\varepsilon^{1/2}\Omega^{3/2} + C\Omega + C\Omega(\varepsilon^2\Omega)^{1/2}|\log\varepsilon|. \tag{5.46}$$

□

Let us now turn to the lower bound to the GP energy with Dirichlet boundary condition. Thereby we find some energy asymptotics.

5.2 Energy Lower Bound and Asymptotics

The energy of the GP functional (3.35) with Neumann boundary conditions is always smaller than the energy under the constraint of Dirichlet boundary conditions. Since lower bounds for the same problem with Neumann boundary conditions were achieved in [CY], we can apply their results.

Theorem 5.1 (Energy Lower Bound with Neumann Boundary Conditions [CY])
As $\varepsilon \to 0$, we have for $|\log\varepsilon| \ll \Omega \lesssim 1/\varepsilon$

$$E^{\mathrm{GP}} \geq E^{\mathrm{TF}} + \frac{\Omega|\log(\varepsilon^2\Omega)|}{2}(1 - o(1)) \tag{5.47}$$

and for $1/\varepsilon \lesssim \Omega \ll (\varepsilon^2|\log\varepsilon|)^{-1}$

$$E^{\mathrm{GP}} \geq E^{\mathrm{TF}} + \frac{\Omega|\log\varepsilon|}{2}(1 - o(1)). \tag{5.48}$$

By recalling (4.1) we find that our method of imposing Dirichlet boundary conditions provides an upper bound that includes a term C/ε for $|\log\varepsilon| \ll \Omega \lesssim 1/\varepsilon$, which partly exceeds $\Omega|\log(\varepsilon^2\Omega)|$. The lower bound with Neumann boundary conditions does not include such a term (5.47). Therefore these lower and upper bounds are not sufficient to reproduce for all Ω in the considered domain a second order expansion of the energy, but a leading order expansion,

$$E^{\mathrm{GP}} = E^{\mathrm{TF}}(1 + o(1)), \tag{5.49}$$

in the regime where $C/\varepsilon \gtrsim \Omega|\log(\varepsilon^2\Omega)|$. If, however, Ω satisfies $(\varepsilon|\log(\varepsilon^2\Omega)|)^{-1} \ll \Omega \lesssim \varepsilon^{-1}$, we have

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \frac{\Omega}{2}|\log(\varepsilon^2\Omega)|(1 + o(1)). \tag{5.50}$$

On the other hand, if we consider the upper bound for $\varepsilon^{-1} \lesssim \Omega \ll \varepsilon^{-2}$ (4.2), we are able to find a second order expansion in the regime where $C\varepsilon^{1/2}\Omega^{3/2}$ is much smaller than $\Omega|\log\varepsilon|$, which is the case for all Ω satisfying $\varepsilon^{-1} \lesssim \Omega \ll \varepsilon^{-1}|\log\varepsilon|^2$. Then we have

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \frac{\Omega}{2}|\log\varepsilon|(1 + o(1)). \tag{5.51}$$

For all Ω satisfying $1/(\varepsilon^2|\log\varepsilon|) \gg \Omega \gtrsim \varepsilon^{-1}|\log\varepsilon|^2$ the term originating from the Dirichlet boundary condition, $C\varepsilon^{1/2}\Omega^{3/2}$, is the second order contribution in our upper bound for the GP energy (4.2) instead of $\Omega/2|\log\varepsilon|$, and therefore the lower bound (5.48) is not sufficient to argue a second order expansion, but we find

$$E^{\mathrm{GP}} = E^{\mathrm{TF}}(1 + o(1)). \tag{5.52}$$

6 A Refined Method to Determine the Energy

This chapter is devoted to refine the methods for a new proof of the energy asymptotics to (3.35). The basic idea is a decoupling of this GP energy functional, which seperates the energy contribution that originates from the occurence of vortices from the contribution of the profile of the Bose-Einstein condensate. After the decoupling both functionals are estimated seperately. Furthermore, we reproduce the energy asymptotics achieved in the previous chapter. Indeed, this method leads in principle to better bounds than the previous one.

6.1 Additive Decoupling of the Energy

In this section we use the notation $\|\cdot\|_p$ as well as $\|\cdot\|_{L_p(\mathcal{D})}$. The first symbol means $\|\cdot\|_p = \|\cdot\|_{L_p(\mathcal{B}_1)}$, while the other points out the restriction to a set $\mathcal{D} \subseteq \mathcal{B}_1$. Let us now start by proving the additive energy decoupling (4.7).

Proof: We assume $\varphi(\vec{r}) > 0$ for $|\vec{r}| < 1$. Then we can write any $\psi \in \mathcal{D}^{\mathrm{GP}}$ as $\psi = \varphi \cdot u$ with u complex-valued. Because of the identity

$$\int |\vec{\nabla}\psi|^2 = -\int |u|^2 \varphi \Delta\varphi + \int \varphi^2 |\vec{\nabla} u|^2 \tag{6.1}$$

the kinetic energy term of the GP functional (3.35) can be written as

$$\begin{aligned}\int |\vec{\nabla}_A \psi|^2 &= \int |\vec{\nabla}\psi|^2 + i\int \vec{A}\psi^* \vec{\nabla}\psi - i\int \vec{A}\psi\vec{\nabla}\psi^* + \int |\vec{A}|^2|\psi|^2 = \\ &= -\int |u|^2\varphi\Delta\varphi - \int \varphi^2|\vec{\nabla}u|^2 + \int \left\{ i\psi^*\vec{A}\varphi\vec{\nabla}u - i\psi\varphi\vec{A}\vec{\nabla}u^* \right\} + \int |\vec{A}|^2|\psi|^2 = \\ &= -\int |u|^2\varphi\Delta\varphi + \int \varphi^2 |\vec{\nabla}_A u|^2. \end{aligned} \tag{6.2}$$

By taking (6.2) into account we rewrite the GP functional (3.35) and get

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \int_{\mathcal{B}_1} \left\{ -|u|^2\varphi\Delta\varphi + \varphi^2|\vec{\nabla}_A u|^2 - \frac{1}{4}r^2\Omega^2|\varphi u|^2 + \frac{1}{\varepsilon^2}|\varphi u|^4 \right\}. \tag{6.3}$$

Now, by taking $\varphi = g$, where g is the (positive) minimizer of (3.43), we may use the GP equation (3.43), together with (3.45) and obtain

$$\begin{aligned}\mathcal{E}^{\mathrm{GP}}[\psi] &= \int_{\mathcal{B}_1} \left\{ |u|^2|g|^2 \left(\frac{1}{4}r^2\Omega^2 - \frac{2}{\varepsilon^2}|g|^2 + \hat{\mu}^{\mathrm{GP}} \right) + |g|^2|\vec{\nabla}_A u|^2 - \frac{1}{4}r^2\Omega^2|gu|^2 + \frac{1}{\varepsilon^2}|gu|^4 \right\} = \\ &= \hat{E}^{\mathrm{GP}} + \int_{\mathcal{B}_1} \left\{ |g|^2|\vec{\nabla}_A u|^2 + \frac{1}{\varepsilon^2}|g|^4(1-|u|^2)|^2 \right\} \equiv \hat{E}^{\mathrm{GP}} + \mathcal{F}_g[u], \end{aligned} \tag{6.4}$$

where we additionally used the mass constraint $\|\psi\|_2 = 1$.

□

We continue our investigation with lower and upper bounds to $\hat{E}^{\mathrm{GP}}$.

6.2 Estimates on $\hat{E}^{\mathrm{GP}}$

The minimum of the functional $\hat{\mathcal{E}}^{\mathrm{GP}}[\varphi]$ is regarded as the energy of the condensate without the contribution of vortices, i.e., the energy of the profile (including the constraint of Dirichlet boundary condition). We begin this section with a simple lower bound to the energy and then state the upper bound.

Proposition 6.1 (Lower Bound to $\hat{E}^{\rm GP}$)
A simple lower bound is given by

$$\hat{E}^{\rm GP} \geq E^{\rm TF}. \tag{6.5}$$

Proof: The statement (6.5) trivially follows by omitting the positive kinetic energy term and the definition of the TF ground state energy.

$$\hat{\mathcal{E}}^{\rm GP}[\varphi] = \int_{\mathcal{B}_1} |\vec{\nabla}\varphi|^2 + \int_{\mathcal{B}_1} \left\{ -\frac{1}{4} r^2 \Omega^2 |\varphi|^2 + \frac{1}{\varepsilon^2} |\varphi|^4 \right\} \geq \mathcal{E}^{\rm TF}[|\varphi|^2] \geq E^{\rm TF} \tag{6.6}$$

□

Proposition 6.2 (Upper Bound to $\hat{E}^{\rm GP}$)
For $\varepsilon \to 0$ and $1 \ll \Omega \lesssim 1/\varepsilon$ we have

$$\hat{E}^{\rm GP} \leq E^{\rm TF} \left(1 + \mathcal{O}(\varepsilon)\right) \tag{6.7}$$

and for $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$

$$\hat{E}^{\rm GP} \leq E^{\rm TF} \left(1 + \mathcal{O}\left(\sqrt{\varepsilon/\Omega}\right) + \mathcal{O}\left(\varepsilon^2 |\log\varepsilon|\right)\right). \tag{6.8}$$

We see, our trial function with Dirichlet boundary condition still implies $E^{\rm TF}$ as leading order contribution in the energy expansion of $\hat{E}^{\rm GP}$, exactly as for Neumann boundary conditions in [CDY]. However, the subsequent order may include the kinetic energy originating from the decrease of density of the Bose-Einstein condensate close to the boundary. Indeed, in our energy upper bound this decrease is represented by the leading relative error. For $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$ the subleading relative error is a consequence of the increasing of the density. In contrast, for slower rotation some additional terms exceeding the term of the increasing of the density are discovered.

Proof: To prove the upper bound to the energy functional (3.42) we test this functional with a wave function of the form

$$\varphi(\vec{r}) = c\sqrt{\rho(\vec{r})}d(\vec{r}) \tag{6.9}$$

where c is the normalization constant and $\rho(\vec{r})$ the regularized TF density (5.11). The factor $d(\vec{r})$ imposes the Dirichlet boundary condition, which is defined by (5.2).

By $\rho^{\rm TF} \leq C(\omega+1)$, the fact that $\rho = \rho^{\rm TF} + \mathcal{O}(\varepsilon^2\Omega)$ and the definition of the regularized density (5.11) we obtain

$$1/c^2 = \int_{\mathcal{B}_1} \rho d^2 = \int_{\mathcal{B}_1} \rho - \int_{\mathcal{I}} \rho(1-d^2) \geq 1 - \varepsilon^2 - C(\omega+1)\delta. \tag{6.10}$$

Hence, the normalization constant satisfies

$$c^2 \leq 1 + C(\omega+1)\delta. \tag{6.11}$$

We estimate the integrand of the kinetic energy contribution in (3.42) by Cauchy-Schwarz inequality,

$$c^2 |(\vec{\nabla}\sqrt{\rho}d)|^2 \leq 2c^2 \left(d^2 |(\vec{\nabla}\sqrt{\rho})|^2 + \rho|(\vec{\nabla}d)|^2\right). \tag{6.12}$$

Now, consider the integral of the first term on the right hand side of (6.12) over $\mathcal{B}_1$ for $\omega > \omega_h$. The definition of the regularized density yields

$$\|d\nabla\sqrt{\rho}\|_2^2 \leq \int_{\mathcal{B}_1} \frac{1}{4}\frac{|\nabla\rho|^2}{\rho} \leq \int_{r<R_h+\Omega^{-1}} \frac{1}{4}\frac{|\nabla\rho|^2}{\rho} + \int_{r\geq R_h+\Omega^{-1}} \frac{1}{4}\frac{|\nabla\rho^{\rm TF}|^2}{\rho^{\rm TF}}. \tag{6.13}$$

By (5.11) and (5.13) the first term of (6.13) is bounded by $C(\varepsilon^2\Omega)\cdot\Omega$. Using the explicit formular of ρ^{TF} we obtain analogously to [CY]

$$\int_{r\geq R_h+\Omega^{-1}} \frac{1}{4}\frac{|\nabla\rho^{\mathrm{TF}}|^2}{\rho^{\mathrm{TF}}} \leq C(\varepsilon^2\Omega)\Omega|\log\varepsilon|. \tag{6.14}$$

The integral of the second term in (6.12) can be estimated by applying $\rho = \rho^{\mathrm{TF}} + \mathcal{O}(\varepsilon^2\Omega)$ and $\rho^{\mathrm{TF}} \leq C(\omega+1)$,

$$\|\vec{\nabla}(d)\sqrt{(\rho)}\|_2^2 = \int_{\mathcal{B}_1} |\vec{\nabla}(d)|^2\rho = \frac{1}{\delta^2}\int_{\mathcal{I}}\rho = \frac{C}{\delta^2}\int_{1-\delta}^{1} r\rho dr \leq \frac{C}{\delta}(\omega+1). \tag{6.15}$$

Alltogether we obtain as upper bound to the kinetic energy term of (3.42)

$$\int_{|\vec{r}|\leq 1} |\vec{\nabla}\varphi|^2 \leq (1+C(\omega+1)\delta)\left(\frac{C}{\delta}(\omega+1) + C(\varepsilon^2\Omega)\Omega|\log\varepsilon|\right). \tag{6.16}$$

To complete our proof it remains to estimate the difference between $\mathcal{E}^{\mathrm{TF}}[\rho^{\mathrm{TF}}] = E^{\mathrm{TF}}$ and $\mathcal{E}^{\mathrm{TF}}[\varphi^2]$. Both terms of the Thomas-Fermi energy functional (3.39) will be considered seperately. At first we consider the nonlinear interaction term. Using (6.11) and $d^2\rho \leq \rho^{\mathrm{TF}}$ we find

$$\begin{aligned} \varepsilon^{-2}\int_{\mathcal{B}_1} d\vec{r}|\varphi|^4 = \varepsilon^{-2}\int_{\mathcal{B}_1} d\vec{r}(c^2d^2\rho)^2 \leq \\ \leq \frac{1+C(\omega+1)\delta}{\varepsilon^2}\int_{\mathcal{B}_1} d\vec{r}(\rho^{\mathrm{TF}})^2 = \varepsilon^{-2}\int_{\mathcal{B}_1} d\vec{r}(\rho^{\mathrm{TF}})^2 + \text{remainder}. \end{aligned} \tag{6.17}$$

If we now use the normalization of ρ^{TF} and $\rho^{\mathrm{TF}} \leq C(\varepsilon\Omega+1)$ we get

$$\text{remainder} \leq C\left((\varepsilon\Omega+1)^2\frac{\delta}{\varepsilon^2}\right). \tag{6.18}$$

By a conversion of the term of the centrifugal contribution given by partial integration and using the normalization of φ we have

$$-\frac{\Omega^2}{4}\int_{\mathcal{B}_1} d\vec{r}r^2|\varphi|^2 = -\frac{\pi\Omega^2}{2} + \pi\Omega^2\int_0^1 drr\Phi(r), \tag{6.19}$$

with

$$\Phi(r) = \int_0^r dr'r'|\varphi(r')|^2. \tag{6.20}$$

Analoguosly one gets

$$-\frac{\Omega^2}{4}\int_{\mathcal{B}_1} d\vec{r}r^2\rho^{\mathrm{TF}} = -\frac{\pi\Omega^2}{2} + \pi\Omega^2\int_0^1 drr\Phi^{\mathrm{TF}}(r), \tag{6.21}$$

with

$$\Phi(r) = \int_0^r dr'r'\rho^{\mathrm{TF}}(r'). \tag{6.22}$$

By the definition of the regularized density (5.11) and using the normalization constant (6.11) we have

$$\Phi(r) \leq \Phi^{\mathrm{TF}}(r) + C(\omega+1)\delta. \tag{6.23}$$

The area of support of Φ and Φ^{TF} has an upper bound $C(\varepsilon\Omega+1)^{-1}$, thus

$$\Omega^2\int_0^1 dr\left\{\Phi^{\mathrm{TF}}(r) - \Phi(r)\right\} \leq C\Omega^2\delta. \tag{6.24}$$

Putting together (6.24) and (6.18) we obtain

$$\mathcal{E}^{\mathrm{TF}}[\varphi^2] - \mathcal{E}^{\mathrm{TF}}[\rho^{\mathrm{TF}}] \leq C\left((\varepsilon\Omega+1)^2\frac{\delta}{\varepsilon^2}\right) + C\Omega^2\delta. \tag{6.25}$$

We now summarize our findings in

$$\hat{E}^{\mathrm{GP}} \leq E^{\mathrm{TF}} + (1+C(\omega+1)\delta)\left(\frac{C}{\delta}(\omega+1) + C(\varepsilon^2\Omega)\Omega|\log\varepsilon|\right) + C(\varepsilon\Omega+1)^2\frac{\delta}{\varepsilon^2}, \tag{6.26}$$

and minimize in terms of δ in two different regimes of rotational velocity. In $1 \ll \Omega \lesssim 1/\varepsilon$ we obtain $\delta \simeq \varepsilon/\sqrt{\varepsilon^2]\log\varepsilon| + 1 + 2\delta}$ and choose $\delta = \varepsilon$. And if we consider $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$ we find a different δ and choose $\delta = \sqrt{\varepsilon/\Omega}$. Both choices satisfy $\delta \ll 1$. Inserting each δ in (6.26) yields the results.

□

6.3 Analysis of the Minimizer g

We now prove some useful statements about the minimizer g, i.e., the unique, positive solution of (3.43). Recall that the uniqueness of g implies spherical symmetry.

Lemma 6.1 (The Minimizer Achieves a Maximum at a Unique Radius)
The minimizer $g(r)$ has only one maximum.

Proof: We remark that in [CRY] they argue similarily, but consider Neumann boundary conditions. We rewrite (3.42) by a variable transformation $r^2 \to s$ and therefore consider g^2 as a function of s, such that the functional $\hat{\mathcal{E}}^{\mathrm{GP}}[g]$ is

$$\pi\int_0^1 \left\{ s|\vec{\nabla}g|^2 - \frac{\Omega^2}{4}sg^2 + \frac{1}{\varepsilon^2}g^4 \right\} ds. \tag{6.27}$$

The mass constraint in the new coordinates is

$$\int_0^1 g^2 ds = 1/\pi. \tag{6.28}$$

First, we note that the variational equation

$$-\Delta g - \frac{\Omega^2 r^2}{4}g + \frac{2}{\varepsilon^2}g^3 = \hat{\mu}^{\mathrm{GP}}g \tag{6.29}$$

implies that g is not constant on any open interval (otherwise $g = 0$ that contradicts the mass constraint $\|g\|_2^2 = 1$). Thus, if $g(r)$ had more than one local maximum, it would have a minimum at some $s = s_2$ with $0 < s_2 < 1$, on the right of a maximum at a position $s = s_1$, i.e., $s_1 < s_2$. Now, for $0 < \varepsilon < g^2(s_1) - g^2(s_2)$ we consider the set $\mathcal{I}_\varepsilon = \{s < s_2 : g^2(s_1) - \varepsilon \leq g^2(s) \leq g^2(s_1)\}$. Since g^2 is continuous, $F(\varepsilon) \equiv \int_{\mathcal{I}_\varepsilon} g^2$ is strictly positive and $F(\varepsilon) \to 0$ for $\varepsilon \to 0$. Likewise, for a $\kappa > 0$ we consider $J_\kappa = \{s > s_1 : g^2(s_2) \leq g^2(s) \leq g^2(s_2) + \kappa\}$. So, the function $G(\kappa) \equiv \int_{J_\kappa} g^2$ has the same properties as F. These properties imply that there exist $\overline{\varepsilon}, \overline{\kappa} > 0$ such that $g^2(s_2) + \overline{\kappa} < g^2(s_1) - \overline{\varepsilon}$ with $F(\overline{\varepsilon}) = G(\overline{\kappa})$. Let us define a new function $\overline{g}^2$ with the same mass constraint as g^2, i.e., (6.28), by

$$\overline{g}^2(s) = \begin{cases} g(s_1)^2 - \overline{\varepsilon} & \text{if } s \in I_{\overline{\varepsilon}} \\ g(s_2)^2 + \overline{\kappa} & \text{if } s \in J_{\overline{\kappa}} \\ g(s)^2 & \text{otherwise} \end{cases} \tag{6.30}$$

The mass constraint is unchanged because what we subtract from g^2 in $I_{\overline{\varepsilon}}$ equals what we add to g^2 in $J_{\overline{\kappa}}$, or rather $F(\overline{\varepsilon}) = G(\overline{\kappa})$. Now, we consider the three terms in (6.27) seperately. The kinetic energy of $\overline{g}^2$ vanishes in the intervalls $I_{\overline{\varepsilon}}$ and $J_{\overline{\kappa}}$, but doesn't differ from g^2 elsewhere. Therefore it is smaller than the kinetic energy by g^2. The potential term for $\overline{g}^2$ is strictly smaller than for g^2 because $-s$ is strictly decreasing and the value of $\overline{g}^2$ on $I_{\overline{\varepsilon}}$ is larger than on $J_{\overline{\kappa}}$. By the definition of $\overline{g}^2$, mass is rearranged from $I_{\overline{\varepsilon}}$ to $J_{\overline{\kappa}}$, where the density is lower, such that $\int \overline{g}^4 < \int g^4$. Thus, the functional evaluated for $\overline{g}^2$ is strictly smaller on $\mathcal{B}_1$. This contradicts the assumption that g^2 is a minimizer. Hence, the minimizer g has no local minimum aside from the boundary points $s = 0$ and $s = 1$.

□

Lemma 6.2 (Upper Bound for the Minimizer g)
If g is a positive function, which satisfies the GP *equation* (3.43)*, we have*

$$\|g\|_\infty^2 \leq \frac{\left(\hat{\mu}^{\mathrm{GP}} + \Omega^2/4\right)\varepsilon^2}{2}. \tag{6.31}$$

Proof: By the fact that $-\Delta g \geq 0$ at the maximum of the positive function g, which we say is located at $r = R$, and (3.43) we obtain the inequality

$$0 \leq \frac{1}{4}R^2\Omega^2 - \frac{2}{\varepsilon^2}g^2(R) + \hat{\mu}^{\mathrm{GP}}, \tag{6.32}$$

which proves our statement.

□

Proposition 6.3 (L^2 Estimates for g)
As $\varepsilon \to 0$, for any Ω with $1 \ll \Omega \lesssim 1/\varepsilon$ we have

$$\|g^2 - \rho^{\mathrm{TF}}\|_2 = \mathcal{O}\left(\sqrt{\varepsilon}\right) \tag{6.33}$$

and for $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$

$$\|g^2 - \rho^{\mathrm{TF}}\|_2 = \mathcal{O}\left(\sqrt{\varepsilon(\varepsilon\Omega)^{3/2}}\right). \tag{6.34}$$

Proof: The structure of the proof is similar as in [CRY]. We start the argumentation with the fact that in the regime $1/\varepsilon \lesssim \Omega$ the TF density is the positive part of some explicit function (A.5). So, by the definition of the TF functional and the upper bound for $\hat{E}^{\mathrm{GP}}$ we find the estimate

$$\begin{aligned}\int_{\mathcal{B}_1} d\vec{r}(g^2 - \rho^{\mathrm{TF}})^2 &= \|g\|_4^4 + \|\rho^{\mathrm{TF}}\|_2^2 - 2\int_{\mathcal{B}_1} d\vec{r} g^2\rho^{\mathrm{TF}} \leq \|g\|_4^4 + \|\rho^{\mathrm{TF}}\|_2^2 - \varepsilon^2\mu^{\mathrm{TF}} - \varepsilon^2\Omega^2\int_{\mathcal{B}_1} d\vec{r} r^2 g^2 \\ &= \varepsilon^2\left(\mathcal{E}^{\mathrm{TF}}[|g|^2] - E^{\mathrm{TF}}\right) \leq \varepsilon^2\left(\hat{E}^{\mathrm{GP}} - E^{\mathrm{TF}}\right) \leq C\varepsilon^2\left(\varepsilon^{1/2}\Omega^{3/2}\right)\end{aligned} \tag{6.35}$$

for any Ω satisfying $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$. By an analogue calculation the estimate for $1 \ll \Omega \lesssim 1/\varepsilon$ is achieved.

□

Lemma 6.3 (Upper Bound for g^2)
For $\varepsilon \to 0$, if Ω satisfies $1 \ll \Omega \lesssim 1/\varepsilon$, we have

$$\|g\|_\infty^2 \leq \rho^{\mathrm{TF}}(1)\left(1 + C\varepsilon^{1/2}\right) \tag{6.36}$$

and, if Ω satisfies $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$, we have

$$\|g\|_\infty^2 \leq \rho^{\mathrm{TF}}(1)(1 + o(1)). \tag{6.37}$$

Proof: The arguments in the proof for $1 \ll \Omega \lesssim 1/\varepsilon$ are similar to those given in [CY]. As a consequence of (6.33), Cauchy-Schwarz inequality and the trivial bound $\|\rho^{\mathrm{TF}}\|_2^2 \leq \|\rho^{\mathrm{TF}}\|_\infty = \rho^{\mathrm{TF}}(1)$ we find

$$\|g^2\|_2^2 - \|\rho^{\mathrm{TF}}\|_2^2 = 2\int_{\mathcal{B}_1} d\vec{r}\rho^{\mathrm{TF}}(g^2-\rho^{\mathrm{TF}}) + \int_{\mathcal{B}_1} d\vec{r}(g^2-\rho^{\mathrm{TF}})^2 \leq C\left(\rho^{\mathrm{TF}}(1)\varepsilon\right)^{1/2}. \tag{6.38}$$

Thus, we get an estimate for the difference of the TF and GP chemical potential by additionally inserting the upper bound for $\hat{E}^{\mathrm{GP}}$,

$$\varepsilon^2(\hat{\mu}^{\mathrm{GP}} - \mu^{\mathrm{TF}}) = \varepsilon^2\left(\hat{E}^{\mathrm{GP}} - E^{\mathrm{TF}}\right) + \|g^2\|_2^2 - \|\rho^{\mathrm{TF}}\|_2^2 \leq C\left(\rho^{\mathrm{TF}}(1)\varepsilon\right)^{1/2}. \tag{6.39}$$

Inserting the variational equation of g (3.43) in

$$-\frac{1}{2}\Delta g^2 \leq -g\Delta g \tag{6.40}$$

together with the explicit formulas for μ^{TF} and ρ^{TF} yields

$$\begin{aligned} -\frac{1}{2}\Delta g^2 &\leq \left(\varepsilon^2\hat{\mu}^{\mathrm{GP}} + \frac{\omega^2}{4} - 2g^2\right)\frac{g^2}{\varepsilon^2} \leq \left(\varepsilon^2(\hat{\mu}^{\mathrm{GP}} - \mu^{\mathrm{TF}}) + 2(\rho^{\mathrm{TF}}(1) - g^2)\right)\frac{g^2}{\varepsilon^2} \leq \\ &\leq 2\left(\left(1 + C\varepsilon^{1/2}\right)\rho^{\mathrm{TF}}(1) - g^2\right)\frac{g^2}{\varepsilon^2}. \end{aligned} \tag{6.41}$$

Since $-\Delta g^2 \geq 0$ at the maximum of g^2 we obtain the first result.

As above we now find for $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$ by composing (6.34), Cauchy-Schwarz inequality and the bound $\|\rho^{\mathrm{TF}}\|_2^2 \leq \|\rho^{\mathrm{TF}}\|_\infty = \rho^{\mathrm{TF}}(1)$ the estimate

$$\begin{aligned} \|g^2\|_2^2 - \|\rho^{\mathrm{TF}}\|_2^2 &= 2\int_{\mathcal{B}_1} d\vec{r}\rho^{\mathrm{TF}}(g^2-\rho^{\mathrm{TF}}) + \int_{\mathcal{B}_1} d\vec{r}(g^2-\rho^{\mathrm{TF}})^2 \leq \\ &\leq \rho^{\mathrm{TF}}(1)^{1/2}\mathcal{O}\left(\varepsilon^{1/2}(\varepsilon\Omega)^{3/4}\right) \leq \rho^{\mathrm{TF}}(1)^{1/2}\mathcal{O}\left(\varepsilon^{1/2}(\varepsilon\Omega)^{3/4}\right). \end{aligned} \tag{6.42}$$

Therefore, we have

$$\varepsilon^2(\hat{\mu}^{\mathrm{GP}} - \mu^{\mathrm{TF}}) = \varepsilon^2\left(\hat{E}^{\mathrm{GP}} - E^{\mathrm{TF}}\right) + \|g^2\|_2^2 - \|\rho^{\mathrm{TF}}\|_2^2 \leq \rho^{\mathrm{TF}}(1)^{1/2}\mathcal{O}\left(\varepsilon^{1/2}(\varepsilon\Omega)^{3/4}\right), \tag{6.43}$$

which by (A.6) implies

$$\hat{\mu}^{\mathrm{GP}} \leq \mu^{\mathrm{TF}} + C\varepsilon^{-1/4}\Omega^{5/4} = \frac{3}{3}\frac{\Omega}{\sqrt{\pi}\varepsilon} - \frac{\Omega^2}{4} + C\varepsilon^{-1/4}\Omega^{5/4}. \tag{6.44}$$

Now we use (6.31), which yields the result.

□

Let us prove a lower bound to the maximum of g, which we will use afterwards in the proof of the lower bound on the position of the maximum.

Proposition 6.4 (Lower Bound for $\sup g^2$**)**
For small enough ε, and if Ω satisfies $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$, we have

$$\|g\|_\infty^2 \geq \frac{1}{3}\frac{\varepsilon\Omega}{\sqrt{\pi}} - C\varepsilon^{7/4}\Omega^{5/4}. \tag{6.45}$$

Proof: By Cauchy-Schwarz inequality and (6.34) we find

$$\|\rho^{\mathrm{TF}}\|_2^2 - \|g^2\|_2^2 = 2\int_{\mathcal{B}_1} d\vec{r}\rho^{\mathrm{TF}}(\rho^{\mathrm{TF}} - g^2) + \int_{\mathcal{B}_1} d\vec{r}(\rho^{\mathrm{TF}} - g^2)^2 \leq$$
$$\leq \rho^{\mathrm{TF}}(1)^{1/2}\mathcal{O}\left(\varepsilon^{1/2}(\varepsilon\Omega)^{3/4}\right) \leq \mathcal{O}\left(\varepsilon^{1/2}(\varepsilon\Omega)^{5/4}\right). \quad (6.46)$$

Composing (6.46) with the equation for the chemical potential $\hat{\mu}^{\mathrm{GP}} = \hat{E}^{\mathrm{GP}} + \varepsilon^{-2}\|g\|_4^4$, the corresponding formula for μ^{TF} and (A.1) together with (A.6) yields

$$\frac{1}{\varepsilon^2}g^2(R) \geq \frac{1}{\varepsilon^2}\|g\|_4^4 = \hat{\mu}^{\mathrm{GP}} - \hat{E}^{\mathrm{GP}} = \mu^{\mathrm{TF}} - E^{\mathrm{TF}} + \varepsilon^{-2}\|g\|_4^4 - \varepsilon^{-2}\|\rho^{\mathrm{TF}}\|_2^2 \geq \frac{1}{3}\frac{\Omega}{\sqrt{\pi\varepsilon}} - C\varepsilon^{-1/4}\Omega^{5/4}, \quad (6.47)$$

and thus we have proved the lower bound.

□

Proposition 6.5 (Lower Bound for the Radius of the Maximum of g)
R denotes the unique radius where the minimizer g attains its maximum. For $\varepsilon \to 0$, and Ω satisfying $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$, we have

$$R^2 \geq 1 - \frac{1}{3}\frac{\omega_h}{\omega} - C\sqrt{\frac{\varepsilon}{\Omega}}. \quad (6.48)$$

Proof: We consider any Ω such that $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$. We estimate the position of the maximum R by composing (3.43), the fact that $-\Delta g \geq 0$ at the maximum and $\hat{\mu}^{\mathrm{GP}} = \hat{E}^{\mathrm{GP}} + \varepsilon^{-2}\|g\|_4^4$, which gives

$$R^2 \geq \frac{4}{\Omega^2}\left(\frac{2}{\varepsilon^2}g^2(R) - \hat{E}^{\mathrm{GP}} - \varepsilon^{-2}\|g\|_4^4\right). \quad (6.49)$$

Now, by (6.8) and the explicit formula for the TF energy, i.e., (A.1), we have

$$\hat{E}^{\mathrm{GP}} \leq E^{\mathrm{TF}} + C\varepsilon^{1/2}\Omega^{3/2} = -\frac{\Omega^2}{4}\left(1 - \frac{8}{3\sqrt{\pi}\omega}\right) + C\varepsilon^{1/2}\Omega^{3/2}. \quad (6.50)$$

Putting together (6.49) and (6.50) results in

$$R^2 \geq 1 + \frac{4}{\Omega^2}\left(\frac{2}{\varepsilon^2}g^2(R) - \frac{2\Omega}{3\sqrt{\pi}\varepsilon} - C\varepsilon^{1/2}\Omega^{3/2} - \varepsilon^{-2}\|g\|_4^4.\right). \quad (6.51)$$

Inserting (6.45) and (6.47) into (6.51) concludes the proof.

□

If the rotational velocity is $\Omega > \omega_h/\varepsilon$, the TF density vanishes on a disc centered at the origin, which we refer to as the hole. Its radius is given by (A.3). We now argue that the minimizer g is at least small in L^2-sense within this area.

Proposition 6.6 (L^2 Estimate for g Inside the Hole)
As $\varepsilon \to 0$, for Ω satisfying $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$, we have

$$\|g\|_{L_2(\mathcal{B}_{R_h})}^2 = \mathcal{O}(\sqrt{\varepsilon}). \quad (6.52)$$

Proof: Our proof is similar to that in [CRY]. We consider Ω such that $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$. Now, we find by the definition (3.42) and the upper bound for $\hat{E}^{\mathrm{GP}}$

$$\int_{\mathcal{B}_1} |\vec{\nabla} g|^2 \leq E^{\mathrm{TF}} - \mathcal{E}^{\mathrm{TF}}[g^2] + C\varepsilon^{1/2}\Omega^{3/2}. \tag{6.53}$$

Since the kinetic energy is positive we obtain

$$\mathcal{E}^{\mathrm{TF}}[g^2] \leq E^{\mathrm{TF}} + C\varepsilon^{1/2}\Omega^{3/2}. \tag{6.54}$$

Observe that for any density ρ normalized to 1 in $L^1(\mathcal{B}_1 \backslash \mathcal{B}_{R_h})$, one has

$$\int_{\mathcal{B}_1 \backslash \mathcal{B}_{R_h}} d\vec{r} \left\{ \varepsilon^{-2}\rho^2 - \Omega^2 r^2 \rho \right\} \geq E^{\mathrm{TF}}. \tag{6.55}$$

Hence, setting $U \equiv g^2$ and denoting by $\|U\|$ the norm in $L^1(\mathcal{B}_1 \backslash \mathcal{B}_{R_h})$, we obtain

$$\mathcal{E}^{\mathrm{TF}}[g^2] \geq E^{\mathrm{TF}}\|U\| + \frac{1}{\varepsilon^2}\left(1 - \frac{1}{\|U\|}\right) \int_{\mathcal{B}_1 \backslash \mathcal{B}_{R_h}} d\vec{r} U^2 - \Omega^2 R_h^2 (1 - \|U\|) \geq$$
$$\geq E^{\mathrm{TF}}\|U\| - \frac{\|U\|(1 - \|U\|)}{\pi\varepsilon^2(1 - R_h^2)} - \Omega^2 R_h^2 (1 - \|U\|) \tag{6.56}$$

by Cauchy-Schwarz inequality. Using the explicit formula for E^{TF} and $R_h^2 = 1 - \omega_h/\omega$ together with the energy bound from above one thus obtains

$$\frac{\Omega}{2\sqrt{\pi}\varepsilon}\left[\|U\|^2 - \frac{7}{3}\|U\| + \frac{4}{3}\right] \leq C\varepsilon^{1/2}\Omega^{3/2} \qquad \text{if } 1/\varepsilon \lesssim \Omega \ll \varepsilon^{-2}. \tag{6.57}$$

Therfore, we have

$$\|U\|^2 - \frac{7}{3}\|U\| + \frac{4}{3} \leq C\varepsilon^{3/2}\Omega^{1/2} \ll C\sqrt{\varepsilon}, \tag{6.58}$$

which implies that $\|U\| \gg 1 - C\sqrt{\varepsilon}$. The statement of the proposition is a consequence of $\|U\|_{L^2(\mathcal{B}_1)} = 1$.

□

Moreover, we now improve the L_2 estimate to the minimizer in a particular subset of the hole to a pointwise estimate of exponential smallness.

Proposition 6.7 (Exponential Smallness of the Minimizer g)
There exists a constant $0 < C < \infty$, such that, for any $\vec{r} \in \mathcal{T}$, whereby

$$\mathcal{T} \equiv \{\vec{r} \in \mathcal{B}_1 : r \leq R_h - \varepsilon\}, \tag{6.59}$$

the minimizer g satisfies for $1/\varepsilon \lesssim \Omega \leq 1/\varepsilon^{4/3}$ the pointwise bound

$$g(r)^2 \leq C\omega \exp\left\{\frac{r^2 - 1}{\sqrt{\varepsilon}|\log\varepsilon|}\right\}, \tag{6.60}$$

and for $1/\varepsilon^{4/3} < \Omega \leq 1/\varepsilon^{7/4}$

$$g(r)^2 \leq C\omega \exp\left\{\frac{r^2 - 1}{\varepsilon^{5/6}|\log\varepsilon|}\right\}, \tag{6.61}$$

and if $1/\varepsilon^{7/4} < \Omega \ll 1/\varepsilon^2$

$$g(r)^2 \leq C\omega \exp\left\{\frac{r^2 - 1}{\varepsilon^{5/4}|\log\varepsilon|}\right\}. \tag{6.62}$$

Proof: Since the argumention is analogue in all proofs we just explain the proof of (6.60). We remark that this proof is similar to that in [CRY]. We start by using the inequality

$$-\frac{1}{2}\Delta U \leq -g\Delta g \tag{6.63}$$

together with the variational equation (3.43) and the fact that

$$\mu^{\mathrm{GP}} - \mu^{\mathrm{TF}} \leq C\varepsilon^{-1/4}\Omega^{5/4} \tag{6.64}$$

and obtain similar to (6.41) the estimate

$$-\frac{1}{2}\Delta U \leq \frac{2}{\varepsilon^2}\left[\frac{\omega^2}{4}(r^2 - R_h^2) + C\varepsilon^{7/4}\Omega^{5/4} - 2U\right]U. \tag{6.65}$$

Let us consider all r for which $r \leq R_h - \varepsilon$. So, we have

$$\frac{\varepsilon^2\Omega^2(-2R_h\varepsilon + \varepsilon^2)}{4} + C\varepsilon^{7/4}\Omega^{5/4} \leq -\frac{R_h\varepsilon^3\Omega^2}{2}(1 - o(1)) \leq -\frac{\varepsilon}{2|\log\varepsilon|} \tag{6.66}$$

and therefore

$$-\frac{1}{2}\Delta U + \frac{1}{\varepsilon|\log\varepsilon|}U \leq 0. \tag{6.67}$$

Consider the function

$$W(r) = \exp\left\{\frac{r^2 - 1}{\sqrt{\varepsilon}|\log\varepsilon|}\right\}, \tag{6.68}$$

which satisfies for any $r \leq 1$

$$-\Delta W + (\varepsilon|\log\varepsilon|)^{-1}W = W(r)\left(-\frac{4r^2}{\varepsilon|\log\varepsilon|^2} - \frac{2}{\sqrt{\varepsilon}|\log\varepsilon|} + \frac{1}{\varepsilon|\log\varepsilon|}\right) \geq 0. \tag{6.69}$$

If we multiply W by $g(R)^2$ we obtain a supersolution to the solution of (6.67), i.e.,

$$g(r)^2 \leq g(R)^2 W(r), \tag{6.70}$$

for any $r \leq R_h - \varepsilon$. The fact $g(R)^2 \leq C\omega$ yields the result.

□

For technical reasons we now define a set where we subsequently find a pointwise estimate to $g(r)$ in terms of ρ^{TF}. Moreover, we consider another set, on which a bound from below to the vortex functional $\mathcal{F}_g[u]$ will be proved later.

Definition 6.1 (Sets of Points Inside the Radius of the Maximum R)
R denotes the unique radius where the minimizer g attains its maximum and R_h is the inner radius at which the TF *density vanishes. So, we define the set*

$$\mathcal{D} \equiv \{\vec{r} \in \mathcal{B}_R : \rho^{\mathrm{TF}} \geq C\omega/|\log\varepsilon|\} \tag{6.71}$$

and

$$\mathcal{D}_s \equiv \{\vec{r} \in \mathcal{B}_R : \rho^{\mathrm{TF}} \geq C\omega/|\log\tau|\} \tag{6.72}$$

with

$$\tau \equiv \varepsilon^2\Omega|\log\varepsilon| \ll 1. \tag{6.73}$$

For $\vec{r} \in \mathcal{D}_s$ we have

$$\rho^{\mathrm{TF}}(\vec{r}) \geq \omega |\log \tau|^{-1} \geq \omega |\log \varepsilon|^{-1} \tag{6.74}$$

since $\tau \gg \varepsilon^2 |\log \varepsilon|^2$, in the regime $|\log \varepsilon| \ll \Omega \ll 1/(\varepsilon^2 |\log \varepsilon|)$, so that

$$0 \geq \log \tau \geq \log(\varepsilon^2 |\log \varepsilon|^2) \geq C \log \varepsilon. \tag{6.75}$$

We remark that $\mathcal{D}$ is, due to the position of the maximum R and the definition of R_h, not empty in the regime where $1/\varepsilon \lesssim \Omega$. The same is true for $1/\varepsilon \gtrsim \Omega$, as the fact that $g^2(R)$ achieves only one maximum together with the exponential smallness of g inside $\mathcal{T}$ and the mass constraint implies $R \geq C > 0$.

Proposition 6.8 (Pointwise Estimate for $g(r)^2$)
If Ω satisfies $1 \ll \Omega \lesssim 1/\varepsilon$ as $\varepsilon \to 0$, then for any $\vec{r} \in \mathcal{D}$ we have

$$|g(r)^2 - \rho^{\mathrm{TF}}(r)| \leq C\varepsilon^{1/2} \tag{6.76}$$

and if $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$, then for any $\vec{r} \in \mathcal{D}$ we have

$$|g(r)^2 - \rho^{\mathrm{TF}}(r)| \leq C\varepsilon^{1/4} \rho^{\mathrm{TF}}(r). \tag{6.77}$$

Proof: We point out the analysis of the case $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$ as comparable arguments are used in our proof of (6.76). Note that our argumentation is similar to that in [AAB]. We start by rewriting the GP equation (3.43) for the minimizer g as

$$-\Delta g = \frac{2}{\varepsilon^2} \left[\tilde{\rho}(r) - |g|^2\right] g \tag{6.78}$$

and thereby define the density

$$\tilde{\rho}(r) \equiv \frac{\varepsilon^2}{2} \left(\frac{r^2 \Omega^2}{4} + \hat{\mu}^{\mathrm{GP}} \right). \tag{6.79}$$

Except of the chemical potential the above density coincides with the TF density ρ^{TF}, see (A.5). By the estimate to the difference between the chemical potentials (6.43) one finds

$$\|\tilde{\rho} - \rho^{\mathrm{TF}}\|_\infty = C\varepsilon^{7/4}\Omega^{5/4}. \tag{6.80}$$

Thus, using the definition of the set $\mathcal{D}$, yields

$$\tilde{\rho}(r) \geq \rho^{\mathrm{TF}}(r)(1 - o(1)) \geq C\omega/|\log \varepsilon|. \tag{6.81}$$

We now find the pointwise estimate by providing suitable super- and subsolutions to (6.78) inside a local interval $r \in [r_0 - \kappa, r_0 + \kappa]$ where $\tilde{R} + \kappa < r_0 < R - \kappa$ with $1 \gg \kappa \geq 0$ and $\tilde{R}$ denoting the inner radius of $\mathcal{D}$. The supersolution has the form

$$W(r) = \sqrt{\tilde{\rho}(r_0 + \kappa)} \coth \left[\coth^{-1} \left(\sqrt{\frac{\tilde{\rho}(R)}{\tilde{\rho}(r_0 + \kappa)}} \right) + \frac{\kappa^2 - |r - r_0|^2}{3\kappa\varepsilon} \sqrt{2\tilde{\rho}(r_0 + \kappa)} \right]. \tag{6.82}$$

In [AS] it was indeed shown that

$$-\Delta W \geq \frac{2}{\varepsilon^2} \left(\tilde{\rho}(r_0 + \kappa) - W^2 \right) W \geq \frac{2}{\varepsilon^2} \left(\tilde{\rho}(r) - W^2 \right) W, \tag{6.83}$$

for any $r \in [r_0 - \kappa, r_0 + \kappa]$, because $\tilde{\rho}$ is increasing in r. At the boundary of the interval the function coincides with the density $\tilde{\rho}$, i.e, $W(r_0 - \kappa) = W(r_0 + \kappa) = \sqrt{\tilde{\rho}(R)}$, which by combining $-\Delta g(R) \geq 0$ and (6.78) is larger than g. Hence, by the maximum principle

$$g(r_0) \leq W(r_0) \leq \sqrt{\tilde{\rho}(r_0+\kappa)}\coth\left[\frac{\kappa}{3\varepsilon}\sqrt{2\tilde{\rho}(r_0+\kappa)}\right] \leq$$
$$\leq \sqrt{\tilde{\rho}(r_0)}\,(1+C\kappa)\left(1+C\exp\left\{-\frac{2\kappa}{3\varepsilon}\sqrt{2\tilde{\rho}(r_0+\kappa)}\right\}\right), \quad (6.84)$$

where we assumed that the argument in coth goes to ∞,

$$\frac{\kappa}{\varepsilon}\sqrt{\tilde{\rho}(r_0+\kappa)} \geq \frac{\kappa}{\varepsilon}\sqrt{\omega/|\log\varepsilon|} \gg 1. \quad (6.85)$$

Simultaneously the first factor in (6.84) has to be $o(1)$. We choose $\kappa = \varepsilon^{1/2}$, such that

$$g(r) \leq \sqrt{\tilde{\rho}(r)}\left(1+C\varepsilon^{1/2}\right), \quad (6.86)$$

for any $\tilde{R}+\kappa < r < R-\kappa$. Since $\tilde{\rho}$ and g are monotonous regarding r one can extend the estimate to $\mathcal{D}$. For example we take a closer look on the extension to the inner radius of $\mathcal{D}$. For any $r \in [\tilde{R}, \tilde{R}+\kappa]$ we have

$$g(r) \leq g(\tilde{R}+\kappa) \leq \sqrt{\tilde{\rho}(\tilde{R}+\kappa)}\left(1+C\varepsilon^{1/2}\right). \quad (6.87)$$

By above estimate (6.84),

$$\sqrt{\tilde{\rho}(\tilde{R}+\kappa)} \leq \sqrt{\tilde{\rho}(\tilde{R})}\,(1+C\kappa)\,. \quad (6.88)$$

So, we conclude that the remainder on the extended domain is the same order. The same arguments hold true for the extension to the outer boundary of $\mathcal{D}$, which closes our proof of the upper bound.

In order to find a lower bound, we fix some r_0 in the intervall $\tilde{R}+\kappa \leq r_0 \leq R-\kappa$. Consider equation (6.78) on the set $r_0 - \kappa \leq r \leq r_0 + \kappa$. Since $\tilde{\rho}$ is an increasing function and g is positive,

$$-\Delta g \geq \frac{2}{\varepsilon^2}\left[\tilde{\rho}(r_0-\kappa) - |g|^2\right] g. \quad (6.89)$$

We find a subsolution to (6.78) by imposing a Dirichlet boundary condition to the same problem on the boundary $\partial\mathcal{B}_R$: Remark that g achieves its maximum on $\partial\mathcal{B}_R$, which is $\mathcal{O}(\omega)$. In [B] it was proven that there is only one positive function h satisfying

$$\begin{cases} -\Delta h = 1/\tilde{\varepsilon}^2[1-h^2]h & \text{in } \Omega \\ h = 0 & \text{on } \partial\Omega \end{cases} \quad (6.90)$$

as $\tilde{\varepsilon} \to 0$ and it is stated that $h \leq 1$. In particular, if we consider the domain $\Omega = \mathcal{B}_R$, it follows from the uniqueness of the positive solution h, that it is radially symmetric. In [Ser] it was proven that there is a lower bound to h, namely

$$1 - C\exp\left\{-\frac{\mathrm{dist}(r,\partial\Omega)}{2\tilde{\varepsilon}}\right\} \leq h. \quad (6.91)$$

Hence, we deduct for $\Omega = \mathcal{B}_R$

$$1 - C\exp\left\{-\frac{R^2-r^2}{2\tilde{\varepsilon}}\right\} \leq h(r) \leq 1. \quad (6.92)$$

If we now define

$$\tilde{h}(r) \equiv \sqrt{\tilde{\rho}(r_0)}h\left(\frac{r-r_0}{\kappa}\right) \quad (6.93)$$

and

$$\tilde{\varepsilon} \equiv \frac{\varepsilon}{\kappa\sqrt{2\rho^{\mathrm{TF}}(r_0)}}, \tag{6.94}$$

then for any $\vec{r} \in \mathcal{B}(r_0 - \kappa, r_0 + \kappa)$ the function $\tilde{h}$ solves

$$-\Delta\tilde{h} = \frac{2}{\varepsilon^2}\left[\tilde{\rho}(r_0 - \kappa) - \tilde{h}^2\right]\tilde{h}, \tag{6.95}$$

with Dirichlet conditions at the boundary $r = r_0 \pm \kappa$. Therefore, $\tilde{h}$ is a subsolution for the problem (6.78), so that by the maximum principle $g(r) \geq \tilde{h}(r)$ and in particular

$$g(r_0) \geq \tilde{h}(r_0) \geq \sqrt{\tilde{\rho}(r_0)}\left[1 - C\exp\left(-\frac{R^2}{2\tilde{\varepsilon}}\right)\right] \tag{6.96}$$

for any $\tilde{R} + \kappa \leq r_0 \leq R - \kappa$. Interchanging $R^2 \geq 1 - C/\omega$ by 1 in (6.96) only changes the constant. Note that in order to have $\tilde{\varepsilon} = o(1)$ the parameter κ has to fulfill the following condition: Since in the regime of our interest we have $\rho^{\mathrm{TF}} \leq C\varepsilon\Omega$ and by the definition (6.94) we find

$$\kappa \gg \sqrt{\frac{\varepsilon}{\Omega}}. \tag{6.97}$$

To extend the domain of the lower bound (6.96) to any $\vec{r} \in \mathcal{D}$ we use the definition of the density $\tilde{\rho}$, i.e., (6.79), the monotonicity of $\tilde{\rho}$ as well as g and (6.81). Thus, we get

$$\tilde{\rho}(r_0 + \kappa) - \tilde{\rho}(r_0) \leq C\varepsilon^2\Omega^2\kappa \leq C\varepsilon\Omega|\log\varepsilon|\kappa\tilde{\rho}(r_0). \tag{6.98}$$

Note, the coefficient of $\tilde{\rho}(r_0)$ is $o(1)$ if

$$\kappa \ll \varepsilon\Omega|\log\varepsilon|. \tag{6.99}$$

To satisfy both conditions on κ, i.e., (6.97) and (6.99), we choose $\kappa = (\varepsilon/\Omega)^{1/2}|\log\varepsilon|^{\gamma}$ with $\gamma > 0$ and therefore find the pointwise lower bound for all $\vec{r} \in \mathcal{D}$

$$g(r) \geq \sqrt{\tilde{\rho}(r)}(1 - C\varepsilon^{1/2}|\log\varepsilon|^{1+\gamma}). \tag{6.100}$$

Alltogether we obtain for the regime $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$ the estimate

$$\tilde{\rho}(r)\left(1 - \mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{1+\gamma})\right) \leq g(r)^2 \leq \tilde{\rho}(r)\left(1 + \mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{1+\gamma})\right) \tag{6.101}$$

for any $\vec{r} \in \mathcal{D}$.

Finally we apply the lower bound for $\tilde{\rho}$, i.e, (6.80) and its abberation from the TF density (6.81), which concludes our proof.

□

Important for the observation of the subleading order effects originating from the zero Dirichlet boundary condition to the energy expansion is the set of points beyond the maximum of the minimizer g.

Definition 6.2 (Set of Points Outside the Radius R)
R denotes the unique radius where the minimizer g attains its maximum. So, we define the set

$$\mathcal{C} \equiv \mathcal{B}_1 \backslash \mathcal{B}_R. \tag{6.102}$$

Let us now introduce the notation $(\mathcal{D}\cup\mathcal{C})^c$ for the complement to the set $\mathcal{D}\cup\mathcal{C}$, where the minimizer is expected to be small.

Proposition 6.9 (L^2 Estimate for g on $(\mathcal{D}\cup\mathcal{C})^c$)
If $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$ we have

$$\|g\|^2_{L^2((\mathcal{D}\cup\mathcal{C})^c)} \leq \frac{C}{|\log\varepsilon|^2}. \tag{6.103}$$

Proof: By the definition of (6.71), $(\mathcal{D}\cup\mathcal{C})^c$ is the set of points for which

$$r^2 < R_h^2 + C(\omega|\log\varepsilon|)^{-1}. \tag{6.104}$$

By the monotonicity of g and the pointwise estimate (6.77) its supremum on $(\mathcal{D}\cup\mathcal{C})^c$ is bounded by

$$\|g\|^2_{L_\infty((\mathcal{D}\cup\mathcal{C})^c)} \leq \rho^{\mathrm{TF}}(\sqrt{R_h + C(\omega|\log\varepsilon|)^{-1}})\left(1 + C\varepsilon^{1/4}\right). \tag{6.105}$$

Thus, we have

$$\int_{(\mathcal{D}\cup\mathcal{C})^c\setminus\mathcal{B}_{R_h}} g^2 \leq \frac{C}{|\log\varepsilon|^2}. \tag{6.106}$$

Since $\|g(r)\|^2_{L_2(\mathcal{B}_{R_h})} = \mathcal{O}(\varepsilon^{1/2})$ (see (6.52)) the result follows.

□

We now compare ρ^{TF} and g^2 on $\mathcal{C}$. This leads to an improved lower bound to the radius where the minimizer attains its maximum.

Lemma 6.4 (Estimate for ρ^{TF} on $\mathcal{C}$)
As $\varepsilon \to 0$ and if Ω satisfies $1 \ll \Omega \lesssim 1/\varepsilon$ we have

$$\left|\|\rho^{\mathrm{TF}} - g\|_{L^1(\mathcal{C})}\right| \leq \mathcal{O}(\varepsilon^{1/2}) \tag{6.107}$$

and if $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$ we have

$$\left|\|\rho^{\mathrm{TF}} - g\|_{L^1(\mathcal{C})}\right| \leq \mathcal{O}\left(|\log\varepsilon|^{-1}\right). \tag{6.108}$$

Proof: Since both proofs follow the same strategy we only prove $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$. Note that the argumentation is symmetric in g^2 and ρ^{TF}. By the norm $\|g\|_2^2 = 1 = \|\rho^{\mathrm{TF}}\|_1$ and the fact that (6.77) on $\mathcal{D}$ we obtain

$$\int_{\mathcal{C}}(\rho^{\mathrm{TF}} - g^2) = \int_{\mathcal{D}}(g^2 - \rho^{\mathrm{TF}}) + \int_{(\mathcal{D}\cup\mathcal{C})^c}(g^2 - \rho^{\mathrm{TF}}) = \mathcal{O}(\varepsilon^{1/4}) + \int_{(\mathcal{D}\cup\mathcal{C})^c}(g^2 - \rho^{\mathrm{TF}}). \tag{6.109}$$

It remains to argue that the second term is smaller than $C\varepsilon^{1/4}$. Inside $\mathcal{B}_{R_h}$ we already achieved the estimate (6.52) and additionally ρ^{TF} vanishes. By (6.103) and the explicit formula for ρ^{TF} we have

$$\left|\int_{(\mathcal{D}\cup\mathcal{C})^c\setminus\mathcal{B}_{R_h}}(g^2 - \rho^{\mathrm{TF}})\right| \leq \frac{C}{|\log\varepsilon|}, \tag{6.110}$$

and thus,

$$\left|\int_{(\mathcal{D}\cup\mathcal{C})^c}(g^2 - \rho^{\mathrm{TF}})\right| \leq \frac{C}{|\log\varepsilon|}. \tag{6.111}$$

By inserting (6.111) in (6.109) the result follows.

□

Proposition 6.10 (Refined Estimate for the Radius R)
If Ω satisfies $1 \ll \Omega \lesssim 1/\varepsilon$ we have

$$R^2 = 1 \pm \mathcal{O}\left(\varepsilon^{-3/2}\Omega^{-2}\right) \tag{6.112}$$

and if $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$ we have

$$R^2 = 1 \pm O\left(|\log\varepsilon|^{-1}\omega^{-2}\right). \tag{6.113}$$

Proof: For the sake of brevity we only consider the case $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$. On the set $\mathcal{C}$, starting from $\rho^{\mathrm{TF}}(R)(1+o(1)) = g^2(R)$ the minimizer g^2 is monotonously decreasing while ρ^{TF} is quadratically increasing in r. Therefore, by (6.108) we have

$$\left|\int_{\mathcal{C}} \frac{\omega^2}{8}(r^2 - R_h^2) - \int_{\mathcal{C}} \frac{\omega^2}{8}(R^2 - R_h^2)(1 + C\varepsilon^{1/2}|\log\varepsilon|)\right| \leq \|\rho^{\mathrm{TF}} - g\|^2_{L^2(\mathcal{C})} \leq \mathcal{O}\left(|\log\varepsilon|^{-1}\right) \tag{6.114}$$

which implies

$$\left|\int_{\mathcal{C}} \frac{\omega^2}{8}(r^2 - R^2)\right| \leq \mathcal{O}\left(|\log\varepsilon|^{-1}\right). \tag{6.115}$$

Performing the integration in (6.115) yields

$$R^2 = 1 \pm \mathcal{O}\left(|\log\varepsilon|^{-1}\omega^{-2}\right) \tag{6.116}$$

which proves our statement. But note that of course $R < 1$.

□

Corollary 6.1 (L_1 Estimate for ρ^{TF} on $\mathcal{C}$)
If Ω satisfies $1 \ll \Omega \ll 1/\varepsilon^2$ we have

$$\|\rho^{\mathrm{TF}}\|_{L_1(\mathcal{C})} \leq o(1). \tag{6.117}$$

Proof: The definition (A.2) together with (6.112) and (6.113) yields the result.

□

We remark that by (6.107) and (6.108) the same as in (6.117) holds true for g^2.

6.4 Upper Bound to the Vortex Functional $\mathcal{F}_g[u]$

This section is devoted to estimate from above the functional $\mathcal{F}_g[u]$, which includes the energy originating from the occurence of vortices.

Proposition 6.11 (Upper Bound to the Vortex Functional)
For $\varepsilon \to 0$ and $1 \ll \Omega \lesssim 1/\varepsilon$ we have

$$\mathcal{F}_g[u] \leq \frac{1}{2}\Omega|\log(\varepsilon^2\Omega)|(1 + o(1)), \tag{6.118}$$

and for $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$,

$$\mathcal{F}_g[u] \leq \frac{1}{2}\Omega|\log\varepsilon|(1 + o(1)). \tag{6.119}$$

Proof: As trial function we take a wave function of the form

$$u = \xi v \tag{6.120}$$

where v is the new symbol for the phase function defined as (5.9) and ξ is the function defined by (5.10), which vanishes at the singularities of v. Remark that the trial function has no factor originating from the Dirichlet boundary condition, since this constraint already has been accounted by the minimizer of $\hat{\mathcal{E}}^{\mathrm{GP}}$ and the product of both functions again fulfills that it is zero at $\partial\mathcal{B}_1$, i.e., the trial function for (3.35) satisfies Dirichlet boundary conditions.

We start with the upper bound to the vortex kinetic energy term in (4.8).

Proposition 6.12 (Vortex Kinetic Energy)
For $\varepsilon \to 0$ and Ω in $1 \ll \Omega \ll 1/\varepsilon^2$ we have

$$\int_{\mathcal{B}_1} |g|^2 |(\vec{\nabla} - i\vec{A})v\xi|^2 \leq \frac{1}{2}\Omega |\log(t^2\Omega)|(1 + o(1)), \tag{6.121}$$

with $\min\{\varepsilon, (\varepsilon\Omega)^{1/2}\} \leq t \ll \Omega^{-1/2}$.

Proof: Corresponding to [CY] we transform the integrand by introducing the electrostatic analogy mentioned in the proof of (5.22). By the facts that v is a phase factor and ξ is real-valued we obtain

$$\begin{aligned}\int_{\mathcal{B}_1} |g|^2 |(\vec{\nabla} - i\vec{A})v\xi|^2 &= \int_{|\zeta-\zeta_i|\leq t} g^2 |\vec{\nabla}(\xi)v|^2 + \int_{\mathcal{B}_1} g^2 |i\xi\vec{\nabla}(v) - i\vec{A}\xi v|^2 = \\ &= \int_{|\zeta-\zeta_i|\leq t} g^2 |\vec{\nabla}\xi|^2 + \int_{\mathcal{B}_1} g^2 |\xi\vec{E}|^2. \end{aligned}\tag{6.122}$$

We turn to the first term in (6.122). Recall that $|\vec{\nabla}\xi| = t^{-1}$ inside each vortex disc $\mathcal{B}_t^i$ and zero in the complement. By inserting the estimate $g^2(R) \leq \rho^{\mathrm{TF}}(1)(1 + o(1))$ inside $\mathcal{D} \cup \mathcal{C}$, which holds for $1 \ll \Omega \ll 1/\varepsilon^2$, we obtain for the first term in (6.122) the estimate

$$\|g\vec{\nabla}\xi\|_2^2 \leq C/t^2 \sum_i \int_{\mathcal{B}_t^i \cap (\mathcal{D}\cup\mathcal{C})} \rho^{\mathrm{TF}}(1)(1 + o(1)) + C/t^2 \sum_i \int_{\mathcal{B}_t^i \cap (\mathcal{D}\cup\mathcal{C})^c} g^2 \tag{6.123}$$

The second term in (6.123) by the exponential smallness of g inside $\mathcal{T}$ and the fact that the number of vortices in the remaining domain is smaller than $C\varepsilon^{-1}$ together with $g^2 \leq C\omega$ bounded by $C\Omega$. Let us now consider the first term in (6.123). By the number of vortices in the support of ρ^{TF}, i.e., $\mathcal{N}' = C\Omega/(1+\omega)$, and $\rho^{\mathrm{TF}} \leq C(\omega + 1)$ we find

$$C/t^2 \sum_i \int_{\mathcal{B}_t^i \cap (\mathcal{D}\cup\mathcal{C})} d\vec{r}\rho^{\mathrm{TF}}(1)(1 + o(1)) \leq C\Omega(1 + \omega)^{-1} \cdot t^2 \cdot t^{-2} \cdot (\omega + 1)(1 + o(1)) = C\Omega. \tag{6.124}$$

In the following we restrict our analysis to all Ω such that $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$ holds, since similar arguments extend the proof to the remaining parameter regime. Note that we now estimate the 'electrostatic term' from above by similar arguments as in the previous section, but additionally use that the ratio $g^2/\rho^{\mathrm{TF}}(r)$ inside $\mathcal{C}$ and $\mathcal{D}$ is smaller than $1 + o(1)$. The value of the upper bound for the ratio is justified by (6.77) and the fact that ρ^{TF} is an increasing function in r while g^2 is decreasing inside $\mathcal{C}$.

$$\begin{aligned}
\int_{(\mathcal{D}\cup\mathcal{C})^c} d\vec{r} g^2(r)\xi^2(r)|\vec{E}(\vec{r})|^2 + \int_{\mathcal{C}\cup\mathcal{D}} d\vec{r}\frac{g^2}{\rho^{\mathrm{TF}}}(r)\cdot\rho^{\mathrm{TF}}(r)\xi^2(r)|\vec{E}(\vec{r})|^2 \leq \\
\leq \int_{(\mathcal{D}\cup\mathcal{C})^c} d\vec{r} g^2(r)\xi^2(r)|\vec{E}(\vec{r})|^2 + \int_{\mathcal{B}_1} d\vec{r}\rho^{\mathrm{TF}}(r)\xi^2(r)|\vec{E}(\vec{r})|^2\cdot(1+o(1)) \leq \\
\leq \int_{(\mathcal{D}\cup\mathcal{C})^c} d\vec{r} g^2(r)\xi^2(r)|\vec{E}(\vec{r})|^2 + \\
+\left(1+\mathcal{O}((t^2\Omega)^{1/2})\right)\sum_i \sup_{\vec{r}\in\mathcal{Q}^i}\rho^{\mathrm{TF}}(\vec{r})\left(\pi|\log(t^2\Omega)|+\mathcal{O}(1)\right)\cdot(1+o(1))
\end{aligned} \tag{6.125}$$

Let us start estimating the first part in (6.125). By the exponential smallness of g^2 inside $\mathcal{T}$ and $g^2(R) = \mathcal{O}(\omega)$ together with the definition of the domain and (5.30) we find

$$\int_{(\mathcal{D}\cup\mathcal{C})^c} d\vec{r} g^2(r)\xi^2(r)|\vec{E}(\vec{r})|^2 \leq \int_{(\mathcal{D}\cup\mathcal{C})^c\setminus\mathcal{T}} d\vec{r} g^2(r)\xi^2(r)|\vec{E}(\vec{r})|^2 + o(1) \leq \mathcal{O}(\Omega). \tag{6.126}$$

The remaining term in (6.125) is estimated as in (5.22), but for the complete argumentation we refer to the paper [CY].

□

We continue our proof of the upper bound to (4.8) by estimating the remaining term. At first, let us consider Ω such that $1 \ll \Omega \lesssim 1/\varepsilon$. Using $\|g\|_\infty^2 \leq C\omega$ and the total number of vortices $\mathcal{N} = \mathcal{O}(\Omega)$ yields

$$\frac{1}{\varepsilon^2}\sum_i \int_{\mathcal{B}_t^i} g^4\left(1-\left(\frac{r}{t}\right)^2\right)^2 \leq \frac{Ct^2}{\varepsilon^2}\cdot\|g\|_\infty^4\cdot\Omega = \frac{Ct^2}{\varepsilon^2}\Omega. \tag{6.127}$$

To finish our proof for $1 \ll \Omega \lesssim 1/\varepsilon$ we set $t = \varepsilon$. Inserting (6.127) and the estimate for the kinetic energy (6.121) in the definition of $\mathcal{F}_g[u]$ yields our result. If Ω satisfies $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$, we follow a different strategy. We split the integration area $\mathcal{B}_1$ of the remaining term in (4.8) into two seperate domains, $\mathcal{D}\cup\mathcal{C}$ and $(\mathcal{D}\cup\mathcal{C})^c$. So we can use that in the former the ratio $g^2/\rho^{\mathrm{TF}}(r)$ is smaller than $1+o(1)$. Now, by the fact $\|\rho^{\mathrm{TF}}\|_\infty \leq C\omega$ we find for the part corresponding to $\mathcal{D}\cup\mathcal{C}$,

$$\begin{aligned}
\sum_i \frac{1}{\varepsilon^2}\int_{\mathcal{B}_t^i\cap(\mathcal{D}\cup\mathcal{C})} (\rho^{\mathrm{TF}})^2\left(1-\left(\frac{r}{t}\right)^2\right)^2(1+o(1)) \leq \sum_i \frac{C\omega}{\varepsilon^2}|\mathcal{Q}_i| \sup_{\vec{r}\in\mathcal{Q}^i}\rho^{\mathrm{TF}}(r_i)\int_{\mathcal{B}_t^i}\left(1-\left(\frac{r}{t}\right)^2\right)^2(1+o(1)) \leq \\
\leq \sum_i \frac{Ct^2\Omega}{\varepsilon}|\mathcal{Q}_i| \sup_{\vec{r}\in\mathcal{Q}^i}\rho^{\mathrm{TF}}(r_i)(1+o(1)) \leq \frac{Ct^2\Omega}{\varepsilon}(1+o(1)),
\end{aligned} \tag{6.128}$$

where we used the normalization of ρ^{TF} and that the Riemann approximation error $\mathcal{R}$ is $o(1)$, i.e., (5.33), in the last line. Since g is exponential small inside $\mathcal{T}$ it remains to consider the contribution originating from $(\mathcal{D}\cup\mathcal{C})^c\setminus\mathcal{T}$. By using that the number of lattice points inside this area is smaller than $\mathcal{O}(\varepsilon^{-1})$ together with $\|g\|_\infty^2 \leq C\omega$ we obtain

$$\frac{1}{\varepsilon^2}\sum_i \int_{\mathcal{B}_t^i\cap((\mathcal{D}\cup\mathcal{C})^c\setminus\mathcal{T})} g^4\left(1-\left(\frac{r}{t}\right)^2\right)^2 \leq C\frac{\Omega^2}{\varepsilon}t^2. \tag{6.129}$$

Now, choosing $t^2 = \varepsilon/\Omega$ yields the result.

□

6.5 Energy Lower Bound

6.5.1 Separation of the Energy

As a main step of our proof of the lower bound to E^{GP} we use that by (4.7) for any $\psi \in H_0^1(\mathcal{B}_1) = \mathcal{D}^{\mathrm{GP}}$ with $\|\psi\|_2 = 1$ the ansatz $\psi = gu$ decouples (3.35) additively into the minimum of a GP type functional (3.42) and (4.8), i.e.,

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \hat{E}^{\mathrm{GP}} + \int_{\mathcal{B}_1} \left\{ \frac{1}{\varepsilon^2}|g|^4(1-|u|^2)^2 + |\vec{\nabla}_A u|^2 |g|^2 \right\}. \tag{6.130}$$

Now, for technical reasons we restrict our consideration of the integral in (6.130) to the domain $\mathcal{D}_s$. This can be done, because of the positivity of the integral kernel. Let us now consider the function

$$u(\vec{r}) \equiv \Psi^{\mathrm{GP}}(\vec{r}) g^{-1}(r) \tag{6.131}$$

such that by (6.130) together with the restriction of the integration domain to $\mathcal{D}_s$ we have

$$E^{\mathrm{GP}} \geq \hat{E}^{\mathrm{GP}} + \int_{\mathcal{D}_s} \left\{ \frac{1}{\varepsilon^2}|g|^4(1-|u|^2)^2 + |\vec{\nabla}_A u|^2 |g|^2 \right\} \equiv \hat{E}^{\mathrm{GP}} + \mathcal{E}_g[u]. \tag{6.132}$$

We remark that we already proved a trivial lower bound to $\hat{E}^{\mathrm{GP}}$ by neglecting the kinetic energy contribution, which leads to E^{TF} (6.5). However, as the minimizer g satisfies an ordinary second order differential equation with Dirichlet boundary condition, one may calculate $\hat{E}^{\mathrm{GP}}$ numerically.

6.5.2 Lower Bound to the Ginzburg-Landau (GL) Type Functional $\mathcal{E}_g[u]$

We begin by stating the lower bound of the functional $\mathcal{E}_g[u]$, which of course is also a lower bound to $\mathcal{F}_g[u]$.

Proposition 6.13 (Lower Bound to the Vortex Functional)
Let u be defined by $\Psi^{\mathrm{GP}} = u \cdot g$. If Ω satisfies $|\log\varepsilon| \ll \Omega \ll 1/(\varepsilon^2|\log\varepsilon|)$ we find for $\varepsilon \to 0$

$$\mathcal{F}_g[u] \geq \mathcal{E}_g[u] \geq \frac{\Omega|\log\gamma|}{2}(1-o(1)), \tag{6.133}$$

where $\gamma \equiv \min[\varepsilon, \varepsilon^2\Omega]$.

Proof: Note that many arguments we may use are analogous to [CY]. Indeed, the difference to their setting at this point is the definition of the domain $\mathcal{D}_s$, which takes care of the fact that we are considering Dirichlet boundary conditions, and $\mathcal{E}_g[u]$ includes the minimizer g^2 instead of ρ^{TF}.

The energy functional $\mathcal{E}_g[u]$ as well as $\mathcal{F}_g[u]$ is a weighted Ginzburg-Landau (GL) type functional, where the Lebesgue measure is replaced by $g^2(r)d\vec{r}$. But there are two more differences that distiguish $\mathcal{F}_g[u]$ from the usual GL setting:

- The internal magnetic field $\vec{A}$ is fixed.
- The coupling parameter is $g^2(\vec{r})\varepsilon^{-2}$, i.e., it depends on the minimizer $g^2(\vec{r})$ at each position.

To handle the dependence of the coupling parameter on the position, we decompose the disc $\mathcal{B}_1$ into small cells arranged on the square regular lattice defined by

$$\hat{\mathcal{L}} \equiv \left\{ \vec{r}_i = (m\hat{l}, n\hat{l}), m, n \in \mathbb{Z} | \hat{\mathcal{Q}}_i \subset \mathcal{D}_s \right\} \tag{6.134}$$

where $\hat{\mathcal{Q}}^i$ is the cell centered at the position $\vec{r}_i$. We also apply the lattice constant from [CY], i.e.,

$$\sqrt{\frac{|\log\varepsilon|}{\Omega}} \ll \hat{l} \ll \min\left[1, \frac{1}{\omega|\log\tau|}\right]. \tag{6.135}$$

Both conditions, (6.73) and (6.135), are consistent, since by multiplying (6.135) with ω and assuming that $\omega \geq |\log\tau|^{-1}$ we find

$$\sqrt{\tau} \ll \omega\hat{l} \ll |\log\tau|^{-1}, \tag{6.136}$$

which can be satisfied, since by definition we have $\tau \ll 1$. In contrast to the smaller cells in the proof for the upper bound, each cell $\hat{\mathcal{Q}}^i$ is now expected to carry a large number of vortices. Now, by the lower bound for the outer radius of $\mathcal{D}_s$ and the definition of the inner radius we find for the width of $\mathcal{D}_s$

$$R^2 - \tilde{R}^2 \geq C/\omega. \tag{6.137}$$

By comparing (6.135) with (6.137) we find that $\hat{l}$ is much smaller, which is useful for extracting the TF profile.

Using the decomposition of $\mathcal{D}_s$ into cells, the pointwise estimate $g^2(r) \geq \rho^{\mathrm{TF}}(r)(1-o(1))$ on $\mathcal{D}_s \subseteq \mathcal{D}$ and the fact that $\rho^{\mathrm{TF}}(r) \geq \rho^{\mathrm{TF}}(r_i)(1-\mathcal{O}(l\omega|\log\tau|))$, which follows from the explicit formula of ρ^{TF} and the lattice size, yields

$$\begin{aligned}\mathcal{E}_g[u] = \int_{\mathcal{D}_s} d\vec{r}|g|^2\left\{\varepsilon^{-2}|g|^2(1-|u|^2)^2 + |u|^2\right\} \geq \\ \geq \sum_{\vec{r}_i\in\hat{\mathcal{L}}}\int_{\hat{\mathcal{Q}}_i}|g|^2\left\{\varepsilon^{-2}|g|^2(1-|u|^2)^2 + |\vec{\nabla}_A u|^2\right\} \geq \\ \geq (1-o(1))\sum_{\vec{r}_i\in\hat{\mathcal{L}}}\rho^{\mathrm{TF}}(r_i)\mathcal{E}^{(i)}[u]\end{aligned} \tag{6.138}$$

with

$$\mathcal{E}^{(i)}[u] \equiv \int_{\mathcal{Q}_i}\left\{\varepsilon^{-2}\rho^{\mathrm{TF}}(r)(1-|u|^2)^2 + |\vec{\nabla}_A u|^2\right\}. \tag{6.139}$$

Correggi and Yngvason [CY] already have achieved a lower bound to $\mathcal{E}^{(i)}[u]$, so we simply state the result and refer to their proof.

Proposition 6.14 (Lower Bound inside Cells)
For any Ω satisfying $|\log\varepsilon| \ll \Omega \ll (\varepsilon^2|\log\varepsilon|)^{-1}$ and ε sufficiently small, it is possible to find l in such a way that (6.135) *is fulfilled and*

$$\mathcal{E}^{(i)}[u] \geq \frac{\Omega l^2|\log\gamma|}{2}(1-o(1)), \tag{6.140}$$

where $\gamma \equiv \min[\varepsilon, \varepsilon^2\Omega]$.

Inserting (6.140) in (6.138) yields

$$\mathcal{E}_g[u] \geq \sum_{\vec{r}_i\in\hat{\mathcal{L}}}\rho^{\mathrm{TF}}(r_i)\frac{\Omega l^2|\log\gamma|}{2}(1-o(1)). \tag{6.141}$$

By the replacement of the Riemann sum by the integral analogously to [CY] (section 5, inequality (5.37)) together with the fact that the contribution to the normalization of ρ^{TF} on the area beyond R is negligible (6.117), i.e.,

$$\int_{\mathcal{C}}\rho^{\mathrm{TF}}(r) = o(1), \tag{6.142}$$

we obtain

$$\mathcal{E}_g[u] \geq \frac{\Omega|\log\gamma|}{2}(1 - o(1)), \tag{6.143}$$

with $\gamma \equiv \min[\varepsilon, \varepsilon^2\Omega]$, which proves our statement.

$\square$

7 Conclusions

Within the 2D GP framework we have proved upper and applied lower bounds to the GP energy of a rapidly rotating Bose-Einstein condensate on a disc with zero Dirichlet boundary condition. By that we could argue that the first order contribution to the energy is the TF energy. In a particular regime we evaluated exactly the subleading contribution due to the occurence of vortices, and found that E^{GP} equals to the second order the energy with Neumann boundaries. By a different approach we could derive upper and lower bounds similar to those in the previous section, but we have now included the kinetic energy of the condensate.

A The TF Energy and Density

The TF ground state energy is explicitly given by

$$\varepsilon^2 E^{\mathrm{TF}} = \begin{cases} \frac{1}{\pi} - \frac{\omega^2}{8} - \frac{\pi\omega^4}{768} & \omega \leq \omega_h \\ -\frac{\omega^2}{4}\left(1 - \frac{8}{3\sqrt{\pi}\omega}\right) & \omega > \omega_h \end{cases} \tag{A.1}$$

Also the unique minimizer can be calculated explicitly. The result is

$$\rho^{\mathrm{TF}} = \begin{cases} \frac{1}{\pi} + \frac{\omega^2}{16} - \frac{\omega^2}{8}(1-r^2) & \omega \leq \omega_h \\ \left[\frac{\omega}{2\sqrt{\pi}} - \frac{\omega^2}{8}(1-r^2)\right]_+ & \omega > \omega_h \end{cases} \tag{A.2}$$

As a consequence of the centrifugal force in the parameter regime $\omega > \omega_h \equiv 4/\sqrt{\pi}$ the density ρ^{TF} responds by creating a hole centered at the origin. Indeed, $\rho^{\mathrm{TF}} = 0$ on a disc with radius R_h defined by

$$R_h \equiv \sqrt{1 - \frac{4}{\sqrt{\pi}\varepsilon\Omega}}. \tag{A.3}$$

In this parameter regime of Ω one then may rewrite the TF minimizer in terms of R_h as

$$\rho^{\mathrm{TF}}(\vec{r}) = \frac{\omega^2}{8}\left[r^2 - R_h^2\right]_+, \tag{A.4}$$

where $[t]_+ = t$, if $t \geq 0$ and 0 otherwise. We can also express the TF density by

$$\rho^{\mathrm{TF}}(\vec{r}) = \frac{1}{2}\left[\varepsilon^2\mu^{\mathrm{TF}} + \omega^2 r^2\right]_+, \tag{A.5}$$

where μ^{TF} is a Lagrangian multiplier determined by the constraint $\|\rho^{\mathrm{TF}}\|_1 = 1$. It is given by

$$\mu^{\mathrm{TF}} = \begin{cases} \frac{2}{\pi\varepsilon^2} - \frac{\Omega^2}{8} & \omega \leq \omega_h \\ -\frac{\Omega^2}{4}\left[1 - \frac{4}{\sqrt{\pi}\omega}\right] & \omega > \omega_h \end{cases} \tag{A.6}$$

Multiplying (A.5) with ρ^{TF} and integrating yields

$$\mu^{\mathrm{TF}} = E^{\mathrm{TF}} + \varepsilon^{-2}\|\rho^{\mathrm{TF}}\|_2^2. \tag{A.7}$$

B Further Analysis of the Minimizer

Since we have refined our statement on the radius of the maximum in (6.113) and (6.112) we straightforwardly find the first order expansion of the maximum of the minimizer.

Corollary B.1 (First Order Expansion of the Maximum of g)
If $1 \ll \Omega \ll 1/\varepsilon^2$ as $\varepsilon \to 0$ we have

$$\|g\|_\infty^2 = \rho^{\mathrm{TF}}(1)(1 + o(1)). \tag{B.1}$$

Proof: Inserting R, i.e., (6.113) and (6.112), in (6.77) yields the statement.

□

Proposition B.1 (Upper Bound for the Gradient of g)
As $\varepsilon \to 0$ for Ω satisfying $1/\varepsilon \lesssim \Omega \ll 1/\varepsilon^2$ we have

$$\|\vec{\nabla} g\|_{L^\infty(\mathcal{D})} \leq C\varepsilon^{1/4} |\log \varepsilon| \Omega \|g\|_{L^\infty(\mathcal{D})}. \tag{B.2}$$

Proof: We use the same strategy as in the corresponding proof of [CRY]. By rewriting the variational equation satisfied by the minimizer g, i.e., (3.43), in polar coordinates and using $g = g(r)$ we find

$$-g'' - \frac{1}{r} g' = \frac{2}{\varepsilon^2} \left(\tilde{\rho} - g^2\right) g. \tag{B.3}$$

Now, taking the $L^\infty(\mathcal{D})$ norm of the equation (B.3) yields

$$\begin{aligned} \|g''\|_{L^\infty(\mathcal{D})} &\leq C\|r^{-1} g'\|_{L^\infty(\mathcal{D})} + \frac{2}{\varepsilon^2} \| \left[\tilde{\rho}(r) - g^2(r)\right] g(r) \|_{L^\infty(\mathcal{D})} \leq \\ &\leq C\|g'\|_{L^\infty(\mathcal{D})} + C\varepsilon^{1/2} |\log \varepsilon|^{1+\gamma} \Omega^2 \|g\|_{L^\infty(\mathcal{D})}, \end{aligned} \tag{B.4}$$

where we used the pointwise estimate (6.101) with $\gamma > 0$ and $\tilde{\rho} \leq C\omega^2$. By composing the Gagliardo-Nirenberg inequality [N] and (B.4) we obtain

$$\begin{aligned} \|g'\|_{L^\infty(\mathcal{D})} &\leq C\|g''\|^{1/2}_{L^\infty(\mathcal{D})} \|g\|^{1/2}_{L^\infty(\mathcal{D})} + C\|g\|_{L^\infty(\mathcal{D})} \leq \\ &\leq \left(C\|g'\|_{L^\infty(\mathcal{D})} + C\varepsilon^{1/2} |\log \varepsilon|^{1+\gamma} \Omega^2 \|g\|_{L^\infty(\mathcal{D})} \right)^{1/2} \|g\|^{1/2}_{L^\infty(\mathcal{D})}, \end{aligned} \tag{B.5}$$

which implies with $\gamma = 1$

$$\|g'\|_{L^\infty(\mathcal{D})} \leq C\varepsilon^{1/4} |\log \varepsilon| \Omega \|g\|_{L^\infty(\mathcal{D})}. \tag{B.6}$$

□

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